Center for Quantitative Economics

Forecasting Market Risk of Portfolios: Copula-Markov Switching Multifractal Approach

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Abstract: This paper proposes a new methodology for modeling and forecasting market risks of portfolios. It is based on a combination of copula functions and Markov switching multifractal (MSM) processes. We assess the performance of the copula-MSM model by computing the value at risk of a portfolio composed of the NASDAQ composite index and the S&P 500. Using the likelihood ratio (LR) test by Christoffersen (1998), the GMM duration-based test by Candelon et al. (2011) and the superior predictive ability (SPA) test by Hansen (2005) we evaluate the predictive ability of the copula-MSM model and compare it to other common approaches such as historical simulation, variance-covariance, Risk-Metrics, copula-GARCH and constant conditional correlation GARCH (CCC-GARCH) models. We find that the copula-MSM model is more robust, provides the best fit and outperforms the other models in terms of forecasting accuracy and VaR prediction.

Keywords *Copula, Multifractal processes, GARCH, VaR, Backtesting, SPA*

JEL classification G17, C02

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1. Introduction

Since the theoretical works of Jorion (1997) and Dowd (1998), value at risk (VaR) has become a standard measure for quantifying market risk of a single asset or an asset portfolio. It is widely used in practice by financial institutions, portfolio managers and regulators. Sophisticated statistical tools and models such as historical simulation or the Monte-Carlo and variance-covariance approaches have been developed in the literature to compute VaR of portfolios. All these models are characterized by a linear dependence structure and normality. On the one hand, these assumptions reduce the complexity associated with the joint distribution of asset returns and facilitate the computations, on the other hand they may lead to inaccurate VaR forecasts. Empirical observations provide convincing evidence that asset returns exhibit volatility clustering, fat tails, tail dependence, asymmetric correlation and multifractality, and it is obvious that these stylized facts cannot be captured by the normality assumption. To obtain accurate VaR estimates, parametric methods based on econometric models for volatility dynamics as well as semi-parametric methods based on extreme value theory (EVT) have been developed in the literature.[1]

In this paper we propose a new modeling approach that is based on a combination of copulas and multifractal processes. Copulas are a tool to describe nonlinear and tail dependent structures of asset returns (cf. Hürlimann, 2004). Although the idea to use copulas[2] for constructing flexible multivariate distributions with different marginal distributions and different dependence structure is still relatively recent, its application in pricing and risk management is widespread,[3] examples include Cherubini and Luciano (2001); Wei et al. (2004); Jondeau and Rockinger (2006); Wu et al. (2006); Palaro and Hotta (2006); Chiou and Tsay (2008); Huang et al. (2009). These studies employ the empirical distribution and/or simple GARCH-type models as margins and obtain reasonable results. Here, we adopt a completely different modeling approach for the marginal distribution, namely the Markov switching multifractal (MSM). Recently proposed

[1] We refer the reader to Dowd (2002); Küster et al. (2006) for a detailed overview of the various VaR approaches and their advantages and disadvantages.

[2] An introduction to unconditional copula theory can be found in Joe (1997) and Nelsen (1999); extensions to conditional copulas are proposed by Patton (2006).

[3] We refer the reader to Embrechts et al. (2002, 2003) for more details on the applications of copulas in finance.

by Calvet and Fisher (2004), MSM processes have demonstrated their ability to reproduce most stylized facts of financial asset returns and, thus, outperform GARCH-type models in terms of modeling and forecasting volatility (Calvet and Fisher, 2004; Lux, 2008; Lux and Morales-Arias, 2010). Our objective is to propose a model that fits the data better than conventional models and produces more accurate estimates of a portfolio's market risk. We compare the forecasting performance of our new model in terms of predicting the market risk of portfolios with different models in the literature via the likelihood ratio test of Christoffersen (1998), the GMM duration-based test of Candelon et al. (2011) and the superior predictive ability test of Hansen (2005).

The rest of the paper is organized as follows. Section 2 reviews the Markov switching multifractal (MSM) volatility model for marginal distributions. In section 3 the univariate marginals are joined by copulas, resulting in the copula-MSM model. Section 4 presents an empirical application. We evaluate the value-at-risk estimates of the copula-MSM model and compare its performance to other models. Finally, Section 5 concludes.

2. Markov switching multifractal (MSM) volatility model

In this section we briefly review the Markov switching multifractal (MSM) volatility model. In general, financial returns, r_t, are modelled as

$$r_t = \mu_t + y_t, \tag{1}$$

where $\mu_t = \mathbb{E}_{t-1}[r_t]$ is the conditional mean return, y_t is the centered return. For simplicity, we assume that the expected return is a constant, and we focus on modeling the centered returns.

The continuous-time Poisson multifractal model of Calvet and Fisher (2001) and its discretized version (Calvet and Fisher, 2004), the Markov switching multifractal (MSM), supply new tools for modeling and forecasting financial volatility. In the MSM, volatility dynamics are driven by a discrete-time Markov-switching process with a large number of states. Centered returns, y_t, are modeled as:

$$y_t = \sigma_t \epsilon_t, \tag{2}$$

where the innovations ϵ_t follow a standard normal distribution ($\epsilon_t \sim N(0,1)$) and instantaneous volatility σ_t^2 is determined by the product of k volatility components or multipliers $M_t = (M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)})$ and a constant scale factor, σ^2,

$$\sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^{(i)}. \tag{3}$$

The volatility components $M_t^{(i)}$ are persistent, non-negative and satisfy $E[M_t^{(i)}] = 1$. Furthermore, the volatility components $M_t^{(1)}, \ldots, M_t^{(k)}$ at time t are assumed to be stochastically independent. At time t, each volatility component $M_t^{(i)}$ changes with probability γ_i depending on its rank within the hierarchy of multipliers and remains unchanged with probability $1 - \gamma_i$. The probabilities are modeled as

$$\gamma_i = 1 - (1 - \gamma_1)^{(b^{i-1})}, \tag{4}$$

with $\gamma_1 \in [0,1]$ the component at the lowest frequency, and parameter $b \in (1, \infty)$. If component $M_t^{(i)}$ changes, we follow Calvet and Fisher (2004) and assume a two-point distribution for the new value with support m_0 and $2 - m_0$ where $0 < m_0 < 1$ is a parameter (thus, guaranteeing an expectation of unity for all $M_t^{(i)}$). Of course, with probability 1/2 the new draw happens to be identical to the old value.

For the MSM model, it is not obvious how to derive the conditional marginal distribution of asset returns. The conditional cdf of y_t is

$$F(y_t|\mathfrak{I}_{t-1}) = \int_{-\infty}^{y_t} f(u_t|\mathfrak{I}_{t-1}) du_t, \tag{5}$$

where $f(y_t|\mathfrak{I}_{t-1})$ is the conditional density given past information $\mathfrak{I}_{t-1}$. It has the following form

$$f(y_t|\mathfrak{I}_{t-1}) = \sum_{j=1}^{n} f(y_t|M_t = m_j) P(M_t = m_j \mid \mathfrak{I}_{t-1}), \tag{6}$$

where $m_1, \ldots, m_n$ are the $n = 2^k$ variations of the volatility components, i.e. from

$m_1 = (m_0, \ldots, m_0)$ to $m_n = (2 - m_0, \ldots, 2 - m_0)$. Under the assumption that the innovations in (2) are i.i.d. $N(0,1)$, the density of return y_t conditional on volatility state M_t is Gaussian,

$$f(y_t|M_t = m_j) = \frac{1}{\sigma h(m_j)}\phi\left(\frac{y_t}{\sigma h(m_j)}\right), \tag{7}$$

where $\phi(.)$ is the standard normal density and $h(m_j) = \sqrt{\prod_{i=1}^{k} m_j^{(i)}}$ with $m_j^{(i)}$ being the i-th element of vector m_j.

Inserting (6) into (5) we obtain

$$\begin{aligned} F(y_t|\Im_{t-1}) &= \int_{-\infty}^{y_t} f(u_t|\Im_{t-1})du_t \\ &= \int_{-\infty}^{y_t} \sum_{j=1}^{n} f(u_t|M_{t-1} = m_j)P(M_{t-1} = m_j|\Im_{t-1})du_t. \end{aligned} \tag{8}$$

The density $f(u_t|M_{t-1} = m_j)$ is Lebesgue integrable. Due to linearity, $F(y_t|\Im_{t-1})$ becomes

$$\begin{aligned} F(y_t|\Im_{t-1}) &= \sum_{j=1}^{n} P(M_{t-1} = m_j|\Im_{t-1}) \int_{-\infty}^{y_t} \left[\sigma h(m_j)\right]^{-1} \phi\left[u_t/\sigma h(m_j)\right] du_t \\ &= \sum_{j=1}^{n} P(M_{t-1} = m_j|\Im_{t-1})\Phi\left(\frac{y_t}{\sigma h(m_j)}\right), \end{aligned} \tag{9}$$

where $\Phi(.)$ is the standard normal cumulative distribution function.

3. Copula-MSM model

The univariate MSM or GARCH models are now linked by copulas. Copulas provide a general and flexible way to describe (conditional) multivariate distributions and, hence, they allow to compute the value at risk of asset portfolios. We first describe how copulas can be utilized to link conditional asset return distributions.

3.1. Joint distributions of asset returns

Let $y_{1,t}$ and $y_{2,t}$ denote the centered returns of two assets and let $r_{1,t} = \mu_1 + y_{1,t}$ and $r_{2,t} = \mu_2 + y_{2,t}$ denote their uncentered returns. Their conditional joint distribution function of the centered returns is

$$F(y_{1,t}, y_{2,t}|\Im_{t-1}) = C(F_1(y_{1,t}|\Im_{t-1}), F_2(y_{2,t}|\Im_{t-1})),$$

and the conditional joint density is

$$f(y_{1,t}, y_{2,t}|\Im_{t-1}) = c(F_1(y_{1,t}|\Im_{t-1}), F_2(y_{2,t}|\Im_{t-1})) \times f_1(y_{1,t}|\Im_{t-1}) f_2(y_{2,t}|\Im_{t-1}),$$

where c is the copula density and f_1 and f_2 are the marginal densities.

The intertemporal dependence only appears in the marginal distributions while the contemporary dependence structure, i.e. the copula, is independent of the information set. Detailed information on common copula functions is provided in the Appendix.

3.2. Calculation of Value-at-Risk

We now calculate the value-at-risk for the copula-MSM model. Define the portfolio return $r_{p,t}$ as

$$r_{p,t} = \pi r_{1,t} + (1-\pi) r_{2,t},$$

where π and $(1-\pi)$ are the portfolio weights. The value at risk of the portfolio, $VaR_t(\alpha)$, is implicitly defined by

$$Pr(r_{p,t} \leq VaR_t(\alpha)|\Im_{t-1}) = \alpha. \tag{10}$$

Rewriting the VaR definition (10), we obtain

$$\begin{aligned} \Pr\left(r_{p,t} \leq VaR_t(\alpha)|\Im_{t-1}\right) &= \Pr\left(\pi r_{1,t} + (1-\pi) r_{2,t} \leq VaR_t(\alpha)|\Im_{t-1}\right) \\ &= \Pr\left(r_{1,t} \leq \frac{VaR_t(\alpha)}{\pi} - \frac{1-\pi}{\pi} r_{2,t}|\Im_{t-1}\right) = \alpha. \end{aligned}$$

Applying Sklar's theorem, it is obvious that

$$
\begin{aligned}
&\Pr\left(r_{p,t} \leq VaR_t(\alpha)|\mathfrak{I}_{t-1}\right) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{z_t} f(u_{1,t}, u_{2,t}|\mathfrak{I}_{t-1}) du_{1,t} du_{2,t} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{z_t} c(F(u_{1,t}|\mathfrak{I}_{t-1}), F(u_{2,t}|\mathfrak{I}_{t-1})) f(u_{1,t}|\mathfrak{I}_{t-1}) f(u_{2,t}|\mathfrak{I}_{t-1}) du_{1,t} du_{2,t} = \alpha,
\end{aligned}
\tag{11}
$$

where

$$z_t = \frac{VaR_t(\alpha)}{\pi} - \frac{1-\pi}{\pi} u_{2,t},$$

and the conditional cdfs and density functions are given by (9) and its derivative. The value-at-risk is calculated by solving (11) numerically.

3.3. Model contestants

The estimation of value-at-risk of portfolios is a standard task in financial econometrics and there is a large number of models at hand. In the empirical illustration we will compare the copula-MSM model to the following conventional models that are used in research and practice.

Copula-GARCH model: The standard GARCH proposed by Bollerslev (1986) is perhaps the most popular and most widely used univariate volatility model. This is due to its simplicity and its capability to capture volatility clusters observed in financial data (Bollerslev et al., 1994). In GARCH(p,q), centered returns are modeled as

$$y_t = \sigma_t \epsilon_t, \tag{12}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{13}$$

where σ_t^2 is the conditional variance. The parameters have to satisfy the following restrictions to ensure positive conditional variances and stationary: $\omega > 0$, $\alpha_i > 0$ for $i = 1, \ldots, p$, $\beta_i > 0$ for $i = 1, \ldots, q$, and $\sum_i \alpha_i + \sum_i \beta_i < 1$. We assume that the innovations, ϵ_t, follow a standard normal distribution (GARCH-n). Hence, the conditional marginal distribution is Gaussian and straightforward to compute.

Historical simulation: The historical simulation is the simplest and most often used VaR model in the literature. This is due to its simplicity to use and calculate. The historical simulation method is free from any assumption and consists in estimating the distribution of the returns via the empirical distribution of the data.

Variance-covariance method: Under the normality assumption, the value-at-risk is

$$VaR_{p,t}(\alpha) = \mu_{p,t} + \sigma_{p,t} Z_\alpha,$$

where $\mu_{p,t}$ and $\sigma_{p,t}$ denote the mean and standard deviation of the portfolio return, and Z_α denotes the α-quantile of the standard normal distribution. The variance $\sigma^2_{p,t}$ is obtained as

$$\sigma^2_{p,t} = [\pi_1 \quad \pi_2] \begin{bmatrix} \sigma^2_{1,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma^2_{2,t} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}.$$

where π_1 and π_2 are the portfolio weights. Typically, the variances and covariances are estimated from historical data.

RiskMetrics method: The RiskMetrics methodology is popular in practice. Under normality the portfolio return distribution is expressed as

$$\sigma^2_{p,t|t-1} = (1 - \lambda) r^2_{p,t-1} + \lambda \sigma^2_{p,t-1|t-2},$$

where $\sigma^2_{p,t|t-1}$ is the estimated variance using data up to a time $t-1$. The parameter λ is constant and known to be $\lambda = 0.94$.

CCC-GARCH method: In the bivariate CCC-GARCH the centered return vector can be modeled as

$$y_t = \Omega_t^{1/2} \epsilon_t \tag{14}$$

where y_t is 2×1 vector of centered returns, Ω_t is the 2×2 variance-covariance matrix, and ϵ_t is the 2×1 vector of innovations in the model. The innovations are assumed to be $N(0, 1)$ distributed and independent. The variance-covariance matrix Ω_t can be decomposed as

$$\Omega_t = D_t \Gamma D_t$$

where D_t is a diagonal matrix and Γ is the correlation matrix. In the bivariate CCC-GARCH(1,1) case

$$\Omega_t = \begin{bmatrix} \sigma^2_{1,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \sigma_{1,t} & 0 \\ 0 & \sigma_{2,t} \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{bmatrix} \begin{bmatrix} \sigma_{1,t} & 0 \\ 0 & \sigma_{2,t} \end{bmatrix}.$$

The advantage of the CCC-GARCH model consists in its simplicity. It allows to disentangle the estimation and prediction of D_{t+1} to obtain $\hat{D}_{t+1}$ and the estimation of Γ resulting in $\hat{\Gamma}$.

We first forecast volatility for each return series in our portfolio, and then use the forecasted volatilities and the estimated correlation to compute the conditional variance of portfolio returns, $\sigma^2_{p,t}$. The VaR forecasts are obtained as

$$VaR_{p,t}(\alpha) = \mu_{p,t} + \sigma_{p,t} Z_\alpha, \tag{15}$$

where Z_α is α-quantile of the standard normal distribution.

4. Empirical Study

4.1. Data

To analyze the performance of the copula-MSM model we consider a portfolio composed of the NASDAQ composite index and the S&P 500 index. The data sets consist of daily closing prices observed between April 15, 2009 and October 12, 2015.[4]

Returns are computed as

$$r_t = 100 \times \ln\left(\frac{P_t}{P_{t-1}}\right), \tag{16}$$

where P_t is the closing value of the index at day t.

Figures 1 and 2 illustrate the time evolution of both stock indices and their returns and squared returns. Volatility clusters are clearly visible in the return and squared return plots. Descriptive statistics of our data sets are reported in

[4]The data for the NASDAQ composite index and the S&P 500 index have been collected from http://research.stlouisfed.org/fred2/series.

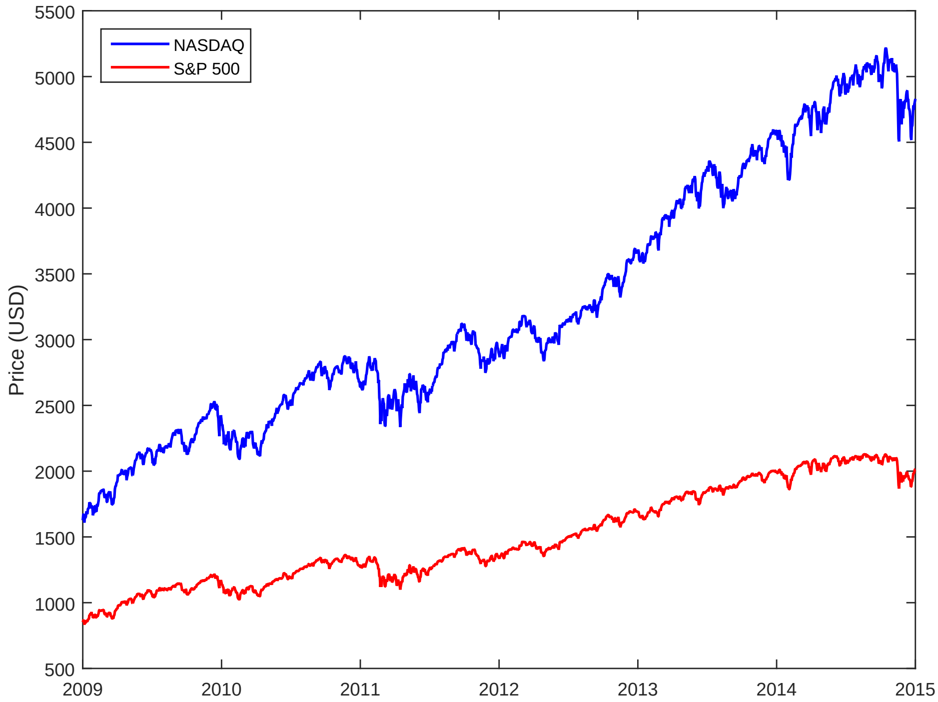

Figure 1: Time evolution of NASDAQ Composite index and S&P 500

Table 1. We observe a moderate unconditional volatility (a daily volatility of 1.1% translates into an annual volatility of about 17%), negative skewness and excess kurtosis in both indices. This is in line with the stylized facts established in many empirical studies. We also compute the Hurst exponents and tail indices for both indices. The values for the Hurst exponent are around 0.5 indicating the absence of long range dependence in returns. The values for the tail index are in the vicinity of 3-4. To gain more insights into the dynamic structure of squared returns, we apply Engle's test for heteroscedasticity. The null hypotheses that the series do not contain ARCH effects are rejected at any usual significance level. The results of the augmented Dickey-Fuller (ADF) unit-root test of Dickey and Fuller (1979) confirm the absence of unit roots in asset returns, and the Jarque-Bera test rejects the null hypothesis of normality.

4.2. Estimation results

We first fit univariate models to the return series. Results for MSM are reported in Table 2, and for GARCH in Table 3. The optimal number of volatility components in the MSM model has been determined by estimating the MSM for all

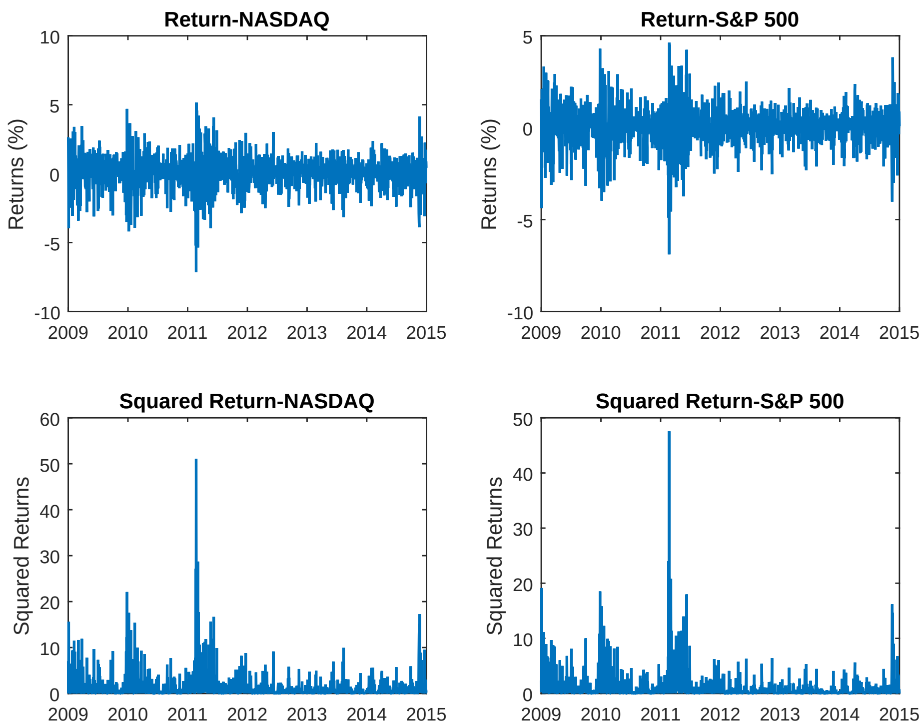

Figure 2: Daily returns and squared returns of NASDAQ Composite index and S&P 500

$k = 1, \ldots, 10$ and comparing their $\log(L)$ values. Results are shown in Table 2. It turns out that 3 or 4 components are appropriate.

The GARCH orders (p, q) have been selected by the Bayesian Information Criterion (BIC) as $p = q = 1$. The GARCH parameters shown in Table 3 are estimated very accurately. The diagnostic tests show that the GARCH models picks up conditional heteroskedasticity and autocorrelation of the NASDAQ index returns, but fail to account for part of the conditional heteroskedasticity of the S&P index.

Having estimated the parameters of the marginal volatility models, we proceed with the estimation of the copula parameters by the inference function for margins (IFM) method. The IFM method consists of two straightforward steps. In the first step, the parameters of the univariate marginal distributions are estimated. Given these estimates, the copula parameters are estimated in the second step. In both steps we use maximum likelihood, hence it is guaranteed that the IFM estimators are asymptotically normal (Joe and Xu, 1996). The estimation results and the model selection criteria are reported in Table 4. The results show

Table 1: Descriptive statistics of stock index returns

Stocks	T	mean	std	skewness	kurtosis	Hurst	tail index	Arch(1)	Arch(5)	Arch(10)	JB	ADF
NASDAQ	1635	0.067	1.139	-0.393	6.009	0.436	3.925	62.257	243.331	292.440	658.200	15.423
								(<0.001)	(0.000)	(0.000)	(0.000)	(0.000)
S&P 500	1635	0.053	1.036	-0.428	6.723	0.435	3.765	69.789	305.699	345.055	993.726	19.948
								(<0.001)	(0.000)	(0.000)	(0.000)	(0.000)

Note: The p-values are reported in parentheses. The values for the tail index are computed by Hill's estimator.

Table 2: Estimation results of MSM model

k	1	2	3	4	5	6	7	8	9	10
					NASDAQ					
$\hat{m}_0$	0.375	0.451	0.488	0.576	0.634	0.668	0.694	0.714	0.730	0.743
$\hat{\sigma}$	1.313	1.344	1.335	1.429	1.190	1.213	1.187	1.176	1.172	1.170
$\hat{b}$	-	16.478	20.936	3.045	1.003	1.938	1.748	1.604	1.514	1.444
$\hat{\gamma}_k$	0.038	0.135	0.913	0.132	0.043	0.195	0.211	0.208	0.210	0.208
log(L)	-2385.203	-2352.973	-2348.761	-2352.018	-2353.449	-2351.511	-2351.511	-2351.485	-2351.491	-2351.476
					S&P					
$\hat{m}_0$	0.300	0.406	0.407	0.450	0.451	0.630	0.409	0.451	0.450	0.411
$\hat{\sigma}$	1.099	1.228	0.974	0.981	2.710	1.092	1.490	1.406	0.609	1.453
$\hat{b}$	-	14.602	14.991	24.803	24.567	1.804	14.919	24.735	24.808	13.184
$\hat{\gamma}_k$	0.071	0.119	0.118	0.932	0.931	0.149	0.118	0.932	0.932	0.115
log(L)	-2182.605	-2135.300	-2136.402	-2134.693	-2135.307	-2136.571	-2136.783	-2135.134	-2136.358	-2137.219

Note: k represents the number of volatility components used in the estimation procedures. $\log L$ is the logarithm of the maximum of the likelihood function. As some estimates $\hat{\gamma}_1$ are very small, we instead report $\hat{\gamma}_k$. Equation (4) can be used for a re-transformation.

that both the Student-t copula-MSM and -GARCH provide the best fit among the copula-MSM models and copula-GARCH models, respectively.

4.3. Evaluating value-at-risk estimates

We analyze the predictive ability of the models using a rolling scheme. Both index price series are divided into two subsets. The first 1135 observations are used as in-sample data for model estimation. The second subset contains the remaining 500 observations and serves as out-of-sample data that we use for forecast evaluation. Note that in the rolling scheme the estimation period is rolled forward by adding one new observation and removing the oldest one each day. Hence, the size of the estimation sample remains fixed over the out-of-sample period. We forecast the one-day ahead 1% and 5% VaR using the copula-MSM model and compare it to: copula-GARCH model, historical simulation, the variance-covariance method, RiskMetrics and the CCC-GARCH model. Figures 3 and 4

Table 3: Estimation results for GARCH(1,1) model

Parameters	NASDAQ	S&P 500
ω	0.043[0.012]	0.034[0.008]
α	0.101[0.020]	0.123[0.023]
β	0.863[0.023]	0.844[0.023]
	Diagnostics	
Arch(1)	1.772(0.183)	4.173(0.041)
Arch(5)	8.933(0.112)	17.003(0.005)
Arch(10)	11.078(0.352)	19.553(0.034)
Q(2)	0.035(0.983)	1.046(0.593)
Q(4)	3.817(0.431)	2.660(0.616)
Q(8)	4.990(0.759)	7.485(0.485)

Note: The numbers in square brackets are standard errors. The values reported in parentheses are the p-values of the statistics. Arch and Q denote Engle's test for residual heteroscedasticity and the Ljung-Box test for residual autocorrelation, respectively.

depict the VaR forecasts for all models at the 5% and 1% confidence level.

To evaluate the performance of the copula-MSM model we first apply the likelihood ratio test proposed by Christoffersen (1998) that is based on the violations process[5], and the GMM duration-based test by Candelon et al. (2011) that makes use of the distributional properties of the duration between two consecutive VaR violations to construct moment conditions that can easily be tested. In fact, under valid VaR forecasts the duration between two consecutive VaR violations follows a geometric distribution with a success of probability equal to the coverage rate (α) (cf. Kupiec, 1995). Recently introduced in the literature, the GMM duration-based test has good power properties for realistic sample sizes compared to LR based tests, cf. Candelon et al. (2011). This is one of the merits of the GMM duration-based test. We conduct both tests for one-period-ahead VaR forecasts. The p-values are reported in Tables 5, 6, 7, and 8. Further, we also apply the superior predictive ability (SPA) test proposed by Hansen (2005) using a VaR-based loss function (see Appendix). The SPA test provides the opportunity to

[5] A violation occurs when ex post portfolio returns are lower than the predicted VaRs.

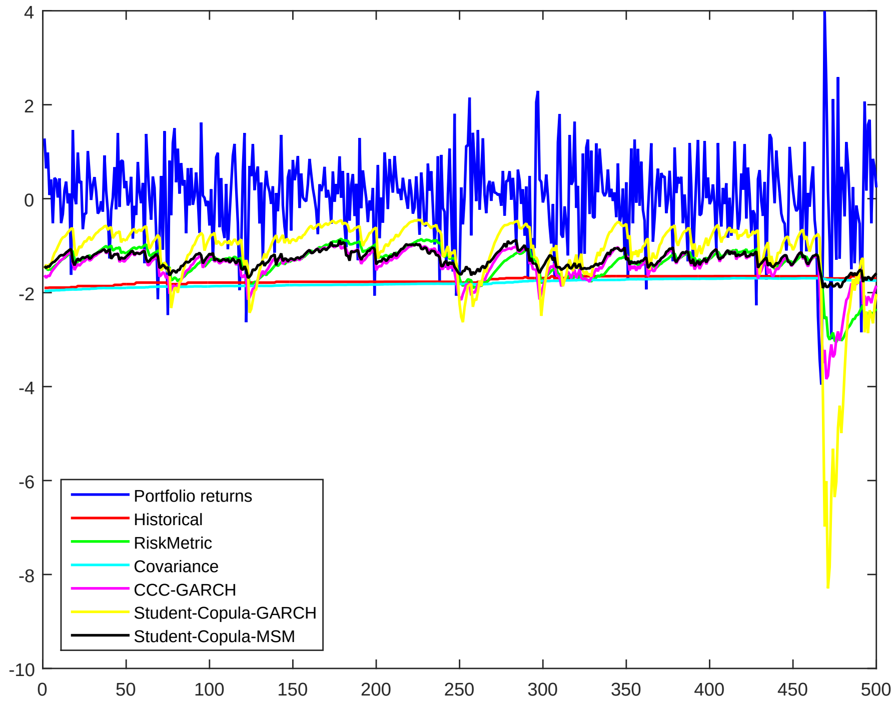

Figure 3: Plot of VaR forecasts at the 5% confidence level

compare the relative forecasting performance of a basis model with its competitors. The p-values of the SPA test are computed based on 5000 bootstrap samples under a VaR-based loss function (see Table 9).

From the results in Tables 5 to 9 we have the following observations:

VaR forecasts with 5% coverage rate:

According to the LR backtests, the null hypothesis that the (unconditional) expected frequency of violations is equal to the coverage rate (5%), is rejected for the variance-covariance method, and all copula-GARCH models. These models exhibit too many VaR violations (the frequency of violations observed over the out-of-sample period is significantly greater than the coverage rate) indicating an underestimation of the true risk level. All the copula-GARCH models can provide VaR forecasts that are independent leading to the non-rejection of the independence hypothesis. This indicates that the violation sequence does not contain clustering structure. As result, it seems that the above mentioned copula-GARCH models can properly model the higher-order dynamics of portfolio returns. The conditional coverage hypothesis is rejected for the covariance method, and all copula-GARCH models at the 5% level. Striking is the fore-

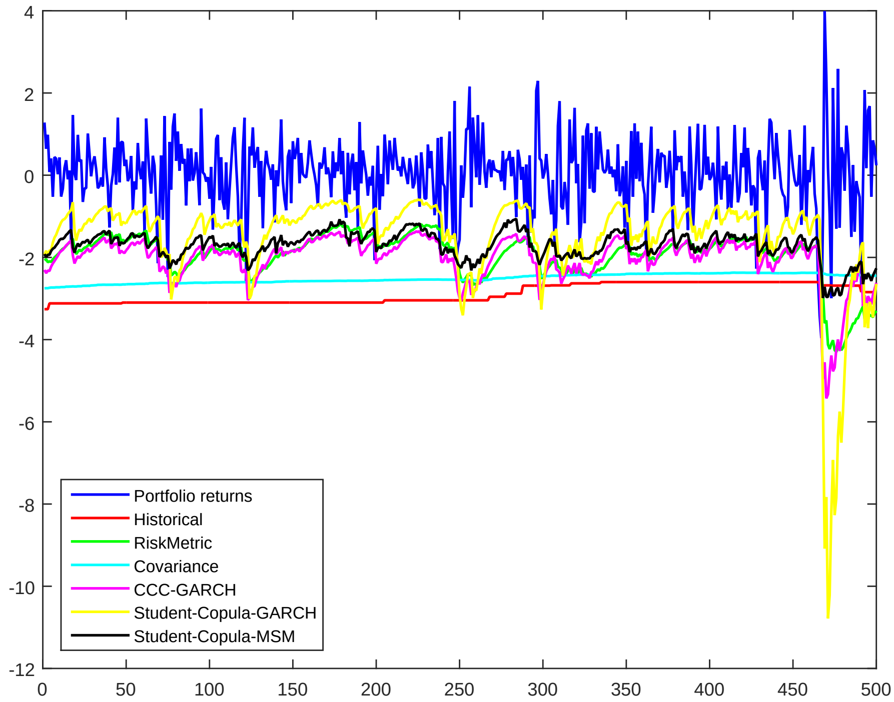

Figure 4: Plot of VaR forecasts at the 1% confidence level

casting performance of the copula-MSM models. All the copula-MSM models used in this paper provide valid VaR forecasts at the 5% level, i.e., the estimated number of violations are not statistically significant different from the coverage rate and the violations are independently distributed. The copula-MSM models outperform the copula-GARCH and the variance-covariance method. However, the historical simulation approach, RiskMetrics and the CCC-GARCH model perform well, too. The GMM duration-based backtesting results match those obtained by the LR backtests. The results of the SPA test using the non-smooth and smooth VaR-based loss functions are almost the same, indicating that a non-differentiable loss function does not impair the implementation of the SPA[6] test procedures. The null hypothesis that a particular model cannot be outperformed by any of the other competitors, is rejected for all copula-GARCH and covariance method at the 5% level.

VaR forecasts with 1% coverage rate:

[6]The SPA test is designed based on the framework of reality check test in White (2000) that requires a differentiable loss function, cf. Theorem 2.3 in White (2000).

According to the LR test results the unconditional coverage hypothesis is rejected at the 1% level for the RiskMetrics, CCC-GARCH, all copula-GARCH and all copula-MSM models. For all these models, the estimated frequency of VaR violations significantly exceeds the coverage rate, indicating an underestimation of the portfolio's actual level of risk. While historical simulation and covariance method cannot produce independent VaR forecasts, the independence hypothesis cannot be rejected for RiskMetrics, CCC-GARCH and for all copula-MSM and copula-GARCH models at the 1% level. These results are confirmed by the GMM duration-based test that properly highlights the capacity and robustness of the copula-MSM models to produce independent and accurate VaR forecasts. While the conditional coverage hypothesis is rejected for all copula-MSM and copula-GARCH models based on the LR results, we observe a non-rejection of the conditional coverage hypothesis as we increase the number of moment conditions for the copula-MSM according to GMM duration-based test results. The independence properties of VaR violations based copula-MSM models and the acceptance of the conditional coverage hypothesis show the capacity of our new model to capture the dynamic structures in higher portfolio return moments. The GMM duration-based test results also reveal the deficiencies of the copula-GARCH models to provide VaR forecasts that fulfill the independence and the unconditional hypotheses, and thus, their incapacity to properly model the higher-order dynamics of portfolio returns. The SPA test results show that all the copula-GARCH models are outperformed by other competitive models at the 1% level.

5. Conclusion

This paper has introduced a new asset portfolio return model that is constructed via a combination of copula functions and Markov-switching multifractal processes (copula-MSM model). We have compared its VaR forecasting ability with those of copula-GARCH models, historical simulation, RiskMetrics, variance-covariance method and CCC-GARCH models. The new model fits the data well and produces accurate and robust VaR forecasts. Its superiority over the copula-GARCH, historical simulation, RiskMetrics, covariance method and the CCC-GARCH model has been documented via LR tests, GMM duration-based tests, and SPA tests. The copula-MSM models is a new tool for value-at-risk calculations that can provide accurate and valid value-at-risk forecasts.

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A. Copula Functions

This paper concentrates on two copula families, namely elliptical copulas (that encompass the Gaussian copula and the Student-t copula), and Archimedean copulas (that include the Clayton copula, rotated Clayton copula, Gumbel copula, rotated Gumbel copula, Frank copula and Plackett copula). In the following we briefly present these copulas. For simplicity, we write $u_t \equiv F_t(x_t)$.

- *Gaussian copula*: The Gaussian copula cumulative distribution function is given by
$$C_g\left(u_1, u_2; \rho\right) = \Phi_\rho\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right), \tag{17}$$
where $0 \le u_1, u_2 \le 1$, Φ_ρ denotes the joint cumulative distribution function of a bivariate normal distribution with correlation $\rho \in (-1, 1)$ and Φ^{-1} is the quantile function of the standard normal distribution.

- *Student-t copula*: In contrast to the Gaussian copula the Student-t copula permits modeling tail dependence, i.e. an increased probability of joint extreme movements. A small value of the degrees-of-freedom parameter ν implies strong tail dependence. This feature makes the Student-t copula attractive for empirical applications. The copula function is
$$C_{st}\left(u_1, u_2; \rho, \nu\right) = t_{\nu,\rho}\left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\right), \tag{18}$$
where $t_{\nu,\rho}$ denotes the bivariate Student-t distribution with ν degrees of freedom and a correlation coefficient ρ, and t_ν^{-1} is the inverse standard univariate Student-t distribution with ν degrees of freedom.

- *Plackett copula*: The Plackett copula is defined as
$$\begin{aligned} C_{pl}\left(u_1, u_2; \varsigma\right) = &\frac{1}{2(\varsigma - 1)}(1 + (\varsigma - 1)(u_1 + u_2) \\ &- \sqrt{(1 + (\varsigma - 1)(u_1 + u_2))^2 - 4\varsigma(\varsigma - 1)u_1u_2}), \end{aligned} \tag{19}$$
where $\varsigma \in [0, \infty)$.

- *Clayton copula and rotated Clayton copula*: The Clayton (1978) copula has an asymmetric dependence structures. The copula functions of the

Clayton copula and the rotated Clayton copula are

$$
\begin{aligned}
C_{cl}(u_1, u_2; \omega) &= \left(u_1^{-\omega} + u_2^{-\omega} - 1\right)^{-\frac{1}{\omega}} \\
C_{r-cl}(u_1, u_2; \omega) &= u_1 + u_2 - 1 + C_{cl}(1 - u_1, 1 - u_2; \omega),
\end{aligned}
\tag{20}
$$

where $\omega \in (0, \infty)$.

- *Symmetrized Joe-Clayton (SJC) copula*: The SJC copula is a modification of the so-called Joe-Clayton copula developed by Joe (1997). It is defined as

$$
\begin{aligned}
C_{sjc}(u_1, u_2; \tau_U, \tau_L) = &\frac{1}{2} - \frac{1}{2}\left[\left([1-(1-u_1)^k]^{-\gamma} + [1-(1-u_2)^k]^{-\gamma} - 1\right)^{-\frac{1}{2}}\right]^{\frac{1}{k}} \\
&\frac{1}{2} - \frac{1}{2}\left[\left([1-u_1^k]^{-\gamma} + [1-u_2^k]^{-\gamma} - 1\right)^{-\frac{1}{2}}\right]^{\frac{1}{k}} + u_1 + u_2 - 1,
\end{aligned}
\tag{21}
$$

with $k = 1/\log 2(2 - \tau_U)$ and $\gamma = -1/\log 2(\tau_L)$ where $\tau_U \in (0, 1)$ and $\tau_L \in (0, 1)$.

- *Frank copula*: The Frank copula is given by

$$
C_{fr}(u_1, u_2; \varrho) = -\frac{1}{\varrho}\ln\left(1 - \frac{(1-\exp(-\varrho u_1))(1-\exp(-\varrho u_2))}{1-\exp(-\varrho)}\right), \tag{22}
$$

where $\varrho \in (-\infty, 0) \cup (0, +\infty)$.

- *Gumbel copula and rotated Gumbel copula*: Like the Clayton copula, the Gumbel copula is asymmetric (Gumbel, 1960), with a higher dependence in the right tails. The Gumbel and rotated Gumbel copula functions are

$$
\begin{aligned}
C_{gu}(u_1, u_2; \delta) &= \exp\left[-\left((-\ln u_1)^{\delta} + (-\ln u_2)^{\delta}\right)^{\frac{1}{\delta}}\right] \\
C_{r-gu}(u_1, u_2; \delta) &= u_1 + u_2 - 1 + C_{gu}(1 - u_1, 1 - u_2; \delta),
\end{aligned}
\tag{23}
$$

where $\delta \in (0, \infty)$

B. Superior Predictive Ability Test

To compare our new model with the traditional VaR models we consider in this study we employ the superior predictive ability test proposed by Hansen (2005). The SPA test is designed based on the framework of White (2000) and allows to compare the relative performance of a baseline model with its competitors via a predefined loss function. Our predefined loss function is the VaR-based loss function introduced in González-Rivera et al. (2004) that is given by

$$\mathrm{VaRl}(\alpha) = T^{-1}\sum_{i=1}^{T}\left(\alpha - I_{t+1}^{\alpha}\right)\left(r_{t+1} - VaR_{t+1}^{\alpha}\right), \tag{24}$$

where T denotes the number of out-of-sample observations, $I_{t+1}^{\alpha} = \mathbf{1}(r_{t+1} < VaR_{t+1}^{\alpha})$, VaR_{t+1}^{α} is the conditional value-at-risk forecast.

To avoid the problems that may occur in the implementation of the superior predictive ability (SPA) test due to the non-differentiability of the VaR-based loss function defined in *eq.* (24) we use a smooth approximation[7] that is given by

$$\mathrm{SVaRl}(\alpha) = T^{-1}\sum_{i=1}^{T}\left[\alpha - h_{\nu}\left(r_{t+1}, VaR_{t+1}^{\alpha}\right)\right]\left(r_{t+1} - VaR_{t+1}^{\alpha}\right), \tag{25}$$

$h_{\nu}(a, b) = [1 + \exp(\nu(a - b))]^{-1}$. The parameter, $\nu > 0$, controls the smoothness and for a higher value of ν $\mathrm{SVaRl}(\alpha)$ gets closer to $\mathrm{VaRl}(\alpha)$ (cf. González-Rivera et al., 2004).

The null hypothesis that the benchmark (or basis) model is not outperformed by any of the other competitive models can be formalized as follows

$$H_0 : \max_{i=1,\ldots,K} \mathbb{E}\left[RP_t\right] \leq 0, \tag{26}$$

where $RP_t = (RP_{i,t}, \ldots, RP_{K,t})'$ is a vector of relative performances, $RP_{i,t}$, that are computed as $RP_{i,t} = VaRL_{t,h}^{(0)} - VaRL_{t,h}^{(i)}$. K is the number of the competitive models, h denotes the forecasting horizon and $VaRL_{t,h}^{(0)}$ and $VaRL_{t,h}^{(i)}$ are the loss functions at time t for a benchmark model M_0 and for its competitor models, $M_{i_{(i=1,\ldots,K)}}$, respectively.

[7] Granger (1999) pointed out that it should always be possible to find a smooth function which is arbitrarily close to the non-smooth one, so that the problem related with the non-differentiability may be just a technical issue.

The associated test statistic is given by

$$T_{SPA} = \max_{i=1,\ldots,K} \frac{\sqrt{T}\bar{RP}_i}{\sqrt{\lim\limits_{T\to\infty} \mathbb{V}ar(\sqrt{T}\bar{RP}_i)}}, \tag{27}$$

where $\bar{RP} = T^{-1}\sum RP_t$. The p-values of the SPA[8] are obtained via a stationary bootstrap procedure using the VaR-based loss function defined above.

[8]More details on technical issues can be found in Hansen (2005).

Table 4: Estimated parameters for different copula functions and model selection criteria.

Copula	Parameter	MSM	GARCH
		NASDAQ-S&P 500	NASDAQ-S&P 500
Gaussian	ρ	0.941	0.946
	Log(L)	1772.538	1836.264
	AIC	-3543.076	-3670.528
	BIC	-3537.677	-3665.129
Student	ρ	0.942	0.946
	ν	7.345	18.606
	Log(L)	**1794.140**	**1840.753**
	AIC	**-3584.279**	**-3677.507**
	BIC	**-3573.482**	**-3666.709**
Plackett	ς	73.702	80.519
	Log(L)	1689.020	1754.584
	AIC	-3376.041	-3507.169
	BIC	-3370.642	-3501.770
Clayton	ω	4.329	4.232
	Log(L)	1415.112	1369.590
	AIC	-2828.224	-2737.181
	BIC	-2822.826	-2731.782
Rotated Clayton	ω	4.682	4.890
	Log(L)	1442.869	1440.392
	AIC	-2883.739	-2878.785
	BIC	-2878.340	-2873.386
SJC	τ_L	0.835	0.843
	τ_U	0.799	0.720
	Log(L)	1724.688	1675.129
	AIC	-3445.375	-3346.258
	BIC	-3434.578	-3335.460
Frank	κ	26.232	25.285
	Log(L)	1336.299	1535.463
	AIC	-2670.599	-3068.925
	BIC	-2665.200	-3063.527
Gumbel	δ	4.341	4.476
	Log(L)	1726.944	1772.624
	AIC	-3451.887	-3543.248
	BIC	-3446.489	-3537.850
Rotated Gumbel	δ	4.219	4.213
	Log(L)	1693.595	1692.211
	AIC	-3385.189	-3382.422
	BIC	-3379.791	-3377.023

Log(L) is the logarithm of the maximum likelihood function. AIC and BIC are the Akaike and Bayesian information criterion, respectively. The volatility components in the MSM is set to 5 ($k = 5$) and the orders in the GARCH $p = q = 1$ that are determined by the Bayesian criterion. The numbers in bold face indicate the maximal Log(L) and minimal AIC and BIC values.

Table 5: The results of LR test using $VaR(5\%)$ forecasts

	Historical	RiskMetrics	Var-Covar	CCC-GARCH					
EFV	0.036	0.066	0.028	0.066					
uc	0.134	0.114	0.015	0.114					
ind	0.155	0.572	0.055	0.894					
cc	0.118	0.245	0.008	0.285					
				Copula-GARCH					
	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
EFV	0.104	0.110	0.094	0.092	0.100	0.088	0.090	0.094	0.092
uc	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
ind	0.464	0.394	0.820	0.897	0.996	0.612	0.548	0.820	0.897
cc	0.000	0.000	0.000	0.001	0.000	0.002	0.001	0.000	0.001
				Copula-MSM					
	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
EFV	0.070	0.070	0.068	0.070	0.070	0.070	0.066	0.070	0.070
uc	0.051	0.051	0.077	0.051	0.051	0.051	0.114	0.051	0.051
ind	0.716	0.716	0.275	0.325	0.325	0.325	0.572	0.325	0.325
cc	0.140	0.140	0.116	0.092	0.092	0.092	0.245	0.092	0.092

Note: EFV denotes the ratio of VaR violations to the sample size ($T = 500$) observed for the portfolio returns. *uc*, *ind* and *cc* denote the p-values related to the unconditional coverage, independence and conditional coverage test statistics, respectively.

Table 6: The results of LR test using $VaR(1\%)$ forecasts

	Historical	RiskMetrics	Var-Covar	CCC-GARCH					
EFV	0.006	0.030	0.012	0.028					
uc	0.334	0.000	0.660	0.001					
ind	0.009	0.073	0.001	0.055					
cc	0.021	0.000	0.004	0.001					
				Copula-GARCH					
	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
EFV	0.076	0.080	0.060	0.058	0.074	0.074	0.064	0.060	0.058
uc	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
ind	0.946	0.899	0.878	0.802	0.869	0.613	0.969	0.878	0.802
cc	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
				Copula-MSM					
	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
EFV	0.040	0.040	0.028	0.024	0.036	0.024	0.028	0.028	0.024
uc	0.000	0.000	0.001	0.008	0.000	0.008	0.001	0.001	0.008
ind	0.234	0.234	0.055	0.028	0.155	0.028	0.055	0.055	0.028
cc	0.000	0.000	0.001	0.003	0.000	0.003	0.001	0.001	0.003

Note: EFV denotes the ratio of VaR violations to the sample size ($T = 500$) observed for the portfolio returns. *uc*, *ind* and *cc* denote the p-values related to the unconditional coverage, independence and conditional coverage test statistics, respectively.

Table 7: The results of GMM duration-based backtesting using $VaR(5\%)$ forecasts

p-values	Historical	RiskMetrics	Var-Covar	CCC-GARCH					
$J_{uc}(1)$	0.173	0.107	0.008	0.111					
$J_{cc}(2)$	0.279	0.183	0.016	0.195					
$J_{cc}(3)$	0.164	0.233	0.012	0.278					
$J_{cc}(4)$	0.101	0.244	0.017	0.342					
$J_{ind}(2)$	0.748	0.899	0.802	0.761					
$J_{ind}(3)$	0.790	0.474	0.491	0.874					
$J_{ind}(4)$	0.487	0.329	0.229	0.893					
				Copula-GARCH					
	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
$J_{uc}(1)$	0.001	0.001	0.002	0.002	0.001	0.003	0.002	0.002	0.002
$J_{cc}(2)$	0.007	0.003	0.012	0.012	0.010	0.017	0.017	0.012	0.012
$J_{cc}(3)$	0.009	0.005	0.017	0.017	0.013	0.023	0.024	0.017	0.016
$J_{cc}(4)$	0.012	0.006	0.021	0.023	0.015	0.033	0.031	0.021	0.022
$J_{ind}(2)$	0.657	0.751	0.271	0.210	0.437	0.620	0.154	0.259	0.210
$J_{ind}(3)$	0.730	0.751	0.396	0.318	0.584	0.773	0.235	0.376	0.310
$J_{ind}(4)$	0.609	0.638	0.496	0.415	0.687	0.890	0.318	0.484	0.406
				Copula-MSM					
	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
$J_{uc}(1)$	0.068	0.062	0.075	0.065	0.064	0.057	0.105	0.059	0.060
$J_{cc}(2)$	0.112	0.108	0.131	0.098	0.100	0.094	0.181	0.102	0.097
$J_{cc}(3)$	0.157	0.151	0.175	0.122	0.136	0.121	0.246	0.138	0.126
$J_{cc}(4)$	0.204	0.199	0.216	0.154	0.172	0.149	0.303	0.177	0.154
$J_{ind}(2)$	0.981	0.978	0.843	0.722	0.843	0.729	0.980	0.842	0.724
$J_{ind}(3)$	0.996	0.995	0.674	0.578	0.808	0.582	0.879	0.816	0.581
$J_{ind}(4)$	0.973	0.973	0.699	0.655	0.886	0.646	0.874	0.890	0.653

J_{uc} represents the unconditional coverage test statistic obtained for $p = 1$. $J_{cc}(p)$ and $J_{ind}(p)$ denote the independence and conditional coverage test statistics based on p moment conditions. The number of moments is fixed to 2,3,4. The Table entries are p-values associated with the GMM duration based test. We note that the GMM duration based test is constructed via moment conditions that are derived from the distribution of durations between consecutive value-at-risk (VaR) violations. Under valid VaR forecasts the duration between two consecutive violations is geometric distributed and the associated orthogonal polynomials are well-known in the literature as a special case of Meixner polynomials. We have used the first 1135 portfolio return observations as in-sample and the remaining 500 observations as out-of-sample.

Table 8: The results of GMM duration-based backtesting using $VaR(1\%)$ forecasts

p-values	Historical	RiskMetrics	Var-Covar	CCC-GARCH
$J_{uc}(1)$	0.510	0.001	0.402	0.000
$J_{cc}(2)$	0.016	0.003	0.137	0.005
$J_{cc}(3)$	0.007	0.007	0.033	0.011
$J_{cc}(4)$	0.006	0.006	0.012	0.012
$J_{ind}(2)$	0.017	0.162	0.002	0.199
$J_{ind}(3)$	0.010	0.378	0.001	0.424
$J_{ind}(4)$	0.005	0.491	0.001	0.519

Copula-GARCH

	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
$J_{uc}(1)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$J_{cc}(2)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$J_{cc}(3)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$J_{cc}(4)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$J_{ind}(2)$	0.020	0.062	0.151	0.910	0.013	0.161	0.011	0.149	0.909
$J_{ind}(3)$	0.094	0.305	0.297	0.920	0.049	0.359	0.023	0.295	0.922
$J_{ind}(4)$	0.219	0.416	0.411	0.955	0.138	0.494	0.055	0.406	0.952

Copula-MSM

	Normal	Student	Plackett	Clayton	rotClayton	SJC	Frank	Gumbel	rotGumbel
$J_{uc}(1)$	0.000	0.000	0.001	0.003	0.000	0.003	0.000	0.001	0.002
$J_{cc}(2)$	0.000	0.000	0.004	0.011	0.000	0.010	0.004	0.004	0.009
$J_{cc}(3)$	0.000	0.000	0.007	0.018	0.001	0.016	0.008	0.008	0.017
$J_{cc}(4)$	0.000	0.000	0.007	0.018	0.000	0.015	0.007	0.008	0.015
$J_{ind}(2)$	0.719	0.713	0.982	0.857	0.844	0.850	0.980	0.980	0.852
$J_{ind}(3)$	0.836	0.834	0.847	0.186	0.876	0.177	0.851	0.848	0.179
$J_{ind}(4)$	0.661	0.661	0.314	0.069	0.504	0.067	0.319	0.314	0.064

J_{uc} represents the unconditional coverage test statistic obtained for $p = 1$. $J_{cc}(p)$ and $J_{ind}(p)$ denote the independence and conditional coverage test statistics based on p moment conditions. The number of moments is fixed to 2,3,4. The Table entries are p-values associated with the GMM duration based test. We note that the GMM duration based test is constructed via moment conditions that are derived from the distribution of durations between consecutive value-at-risk (VaR) violations. Under valid VaR forecasts the duration between two consecutive violations is geometric distributed and the associated orthogonal polynomials are well-known in the literature as a special case of Meixner polynomials. We have used the first 1135 portfolio return observations as in-sample and the remaining 500 observations as out-of-sample.

Table 9: SPA test results of models' comparison

Basis Model	$VaRl(5\%)$	$SVaRl(5\%)$	$VaRl(1\%)$	$SVaRl(1\%)$
Hist	0.298	0.294	0.381	0.372
RiskMetrics	0.299	0.302	0.077	0.078
Covariance	0.000	0.000	0.552	0.549
CCC-GARCH	0.799	0.796	0.821	0.822
Normal-GARCH	0.001	0.001	0.000	0.000
Student-GARCH	0.001	0.001	0.000	0.000
Plackett-GARCH	0.000	0.000	0.002	0.001
Clayton-GARCH	0.000	0.000	0.001	0.001
rotClayton-GARCH	0.001	0.001	0.001	0.001
SJC-GARCH	0.000	0.000	0.000	0.000
Frank-GARCH	0.000	0.000	0.001	0.002
Gumbel-GARCH	0.000	0.000	0.001	0.001
rotGumbel-GARCH	0.000	0.000	0.002	0.002
Normal-MSM	0.349	0.333	0.018	0.019
Student-MSM	0.310	0.300	0.017	0.019
Plackett-MSM	0.641	0.581	0.079	0.084
Clayton-MSM	0.655	0.629	0.591	0.592
rotClayton-MSM	0.277	0.270	0.046	0.050
SJC-MSM	0.195	0.200	0.086	0.090
Frank-MSM	0.987	0.979	0.096	0.102
Gumbel-MSM	0.401	0.347	0.073	0.079
rotGumbel-MSM	0.129	0.125	0.077	0.080

Note: The numbers reported in the Table are the p-values of the SPA test of Hansen (2005) under the null that a basis model cannot be outperformed by other competing models. The values in bold face represent the p-values that are smaller than or equal to the 5% and 1% confidence level under a VaR-based loss function (cf. appendix). Our VaR-based loss function is not differentiable and this may affect the SPA test results. To solve the problem we follow Granger (1999) and use a smooth VaR-based loss function SVaRl(α) which is arbitrarily close to the non-smooth one. We have used the first 1135 portfolio return observations as in-sample and the remaining 500 observations as out-of-sample.