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Should We Like it? – A Social Welfare Based Quantification of Policy Attractiveness

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Should We Like it? - A Social Welfare Based Quantification of Policy Attractiveness*

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Abstract Inequality has again become prominent in public and scientific discussion. There is a vast literature on its measurement, and numerous empirical papers describe and compare the situation in different countries. However, these results can barely be used to evaluate (potential) policy decisions that change the underlying state variables such as income or wealth. A classic approach for such evaluation uses social welfare functions, but the research in this field often only provides conceptional procedures which are not appropriate for empirical comparative studies.

In this paper, we propose a new tool that measures the attractiveness of a policy decision, based on a social welfare function. The so-called *substantial welfare ratio*, is motivated by the literature on performance measurement and incorporates an inherent robustness check. In particular, it investigates the impact of small modifications at the tails of the change distribution. Additionally, it meets several criteria for good inequality and performance measures, making decisions based on the ratio consistent with those based on such measures. We provide an application to European data, for which we adopt a stylized life-cycle model, in order to motivate the shape of the underlying welfare function. The example shows that the new tool can be used to analyse policy decisions such as tax-reforms and thereby reveals differences in the attractiveness and welfare structure.

* This paper uses data from the Eurosystem Household Finance and Consumption Survey.

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1 Introduction

The public and scientific debate on inequality has again become prominent in public and scientific discussion, through such publications as Stiglitz (2012), Bradley (2015) or Dorling (2015). There is a large number of inequality measures which serve various purposes. Most common are the indices from Gini, Atkinson, Dalton, Herfindahl or Theil's entropy. Their properties are well known and the set of potential alternative inequality measures has been well investigated and categorized, e.g. in Cowell (2011). Whereas the development and characteristics of inequality, poverty and richness is extensively documented, including Piketty and Saez (2003), Piketty and Zucman (2014), Kuznets (1955) or Alvaredo and Saez (2009) amongst others, the resulting policy implications are ambiguous (Grodner and Kniesner (2008)). Evaluating policy decisions based on changes in inequality measures depends crucially on the underlying distribution. Accounting for the distribution, as in Chauvel (2016) or Peichl et al. (2010), resolves parts of the problem, but as shown in Atkinson (1970), the implications of changes in inequality measures do not necessarily apply to social welfare (SW) considerations.

Instead, the SW function itself has to be accounted for, because "*the economist (...) is primarily interested, not in the distribution of income as such, but in the effects of the distribution of income upon the distribution and total amount of economic welfare, which may be derived from income*" (Dalton (1920)). Examining the welfare implications of policy decision requires an appropriate measurement of SW, while taking into account the distribution of the state variables. This is of particular importance in order to understand the driving forces behind changes in SW, but is nevertheless completely unattended by standard SW approaches. These evaluations of policy attractiveness subsume the effects on the population, and thus are at odds with the reality of policy makers, who are generally concerned with specific parts of the population. Furthermore, the SW approach does not provide clear policy implications as *absolute* changes in SW do not allow for meaningful interpretation or comparison of policies.

Thus, the impetus for developing our new tool derives from two yet unanswered questions:

- Is there a meaningful way to include policy effects on relevant state variables into welfare considerations and condense it to a normed and intuitive number?
- How much does the computed measure depend on the welfare change of few extreme changes, i.e. on the tails of the gains and losses?

In this paper, we answer these questions and fill the need for an adequate quantification by proposing a SW-based measure of (policy) attractiveness - the *substantial welfare ratio*. Our approach is motivated by the gain loss ratio (GLR) as in Bernardo and Ledoit (2000), which is widely known in the context of performance measurement. More recently the concept was successfully applied in the context of decision making and De Langhe and Puntoni (2015) explain how this approach leads to more desirable decision, partly reproducing properties of prospect theory. Additionally, we adapt the idea of the substantial gain loss ratio of Voelzke (2015), which is a further development of the GLR.

The latter enables us to carry out robustness checks by isolating the *substantial* part of a detected effect. In particular, this allows us to assess which groups of the population are driving the effects and if this is in line with the intentions leading to policy decisions. The substantial welfare ratio meets several criteria, such as consistency with Lorenz dominance or the weak principle of transfer, making it coherent with the general understanding of inequality. The definition is based on arbitrary SW functions. Thus, our approach is generally applicable, because either full model specifications[1] or ad-hoc welfare functions[2] can be adopted. The developed tool is easily applied to several additional state variables, such as health, happiness or unemployment, as proposed by e.g. Fleurbaey (2015), Aronsson (2010), Ng (2003) and Dutta and Foster (2013). Furthermore, we can conveniently depart from the Weisbrod and Hansen (1968) annuity approach of combining income and wealth in a single state variable. Headey and Wooden (2004), Brandolini et al. (2010) and Azpitarte (2012) provide a sound reasoning for the latter. In order to motivate the general form of a welfare function, we adopt a stylized life-cycle model with the two state variables of income and wealth. Matching these features of the model, we use the variables of income, wealth and expected remaining working life from HFCS data to show how the proposed procedure can conveniently be used for policy analysis. In particular, we use the substantial welfare ratio to analyse two hypothetical tax-reforms in four European countries.

The paper proceeds as follows. In Section 2, we introduce the substantial welfare ratio and its properties. In Section 3, we apply the proposed tool to analyse hypothetical tax-reforms for four European countries, using a stylized life-cycle model of consumption and saving. Section 4 concludes.

2 Substantial Welfare Ratio

Consider a reasonable welfare function $\mathrm{SW}(X)$ of the state matrix X. For the following definition, we assume that such a welfare function is the sum of individual value functions $V_i(X_{i,\cdot})$ in some state variables, i.e.

$$\mathrm{SW}(X) := \sum\nolimits_{i \in I} V_i(X_{i,\cdot}), \tag{1}$$

where X is matrix of real numbers with $|I|$ rows and columns equal to the number of relevant state variables. The state variables could be individual wealth, income, leisure time, health and anything that influences the welfare of a population I and can be investigated empirically.

As a measure of attractiveness, the welfare ratio evaluates the change in SW, e.g. induced by a reform or as a comparison between two alternatives.

[1] See Krusell and Smith (1998), Hotchkiss et al. (2012), Quadrini and Ríos-Rull (1997), Huggett (1996) or Castañeda et al. (2003).

[2] See Bergson (1938), Samuelson (1948), Fleurbaey (2009) or Ng (1983).

Definition 1 (Welfare ratio):
Given a SW function as in (1) and two matrices X and X' of state variables, the *welfare ratio* is defined as

$$\mathrm{WR}(X, X') := \frac{\sum_{i\in I}(\Delta V_i)^+}{\sum_{i\in I}(\Delta V_i)^-}, \tag{2}$$

where $\Delta V_i := V_i(X'_{i,\cdot}) - V_i(X_{i,\cdot})$. Further, $(\cdot)^+$ and $(\cdot)^-$ denote the absolute value of the positive and the negative elements respectively. We set the welfare ratio to infinity if $\sum_{i\in I}(\Delta V_i)^- = 0$ and $\sum_{i\in I}(\Delta V_i)^+ > 0$. Additionally, we define the welfare ratio as unity for $\sum_{i\in I}(\Delta V_i)^- = 0$ and $\sum_{i\in I}(\Delta V_i)^+ = 0$, yielding analytic continuation.

Given Definition 1, a welfare ratio greater or equal than one implies that the overall SW is increased, while the absolute value is a measure of attractiveness. The limiting case, a Pareto improvement, results in a welfare ratio equal to infinity, i.e. labelling such a development as super attractive.

For a short example assume that we have a population of two individuals and X is one dimensional. Further assume that $V_1(\cdot) = V_2(\cdot) := \sqrt{\cdot}$. Evaluating a reform that implies a change from $X = (1, 10)$ to $X' = (0, 20)$ results in $\mathrm{WR} = \frac{\sqrt{20}-\sqrt{10}}{\sqrt{1}-\sqrt{0}} \approx 1.3$, i.e. such a change is labelled attractive.

Basing a decision on the welfare ratio, results by construction in conducting reforms which increase implied SW. Additionally, the welfare ratio yields a normalized coefficient enabling us to compare the attractiveness of policies/reforms in different subsets, e.g. partitioned by region or educational background.

However note that comparing competing reforms, e.g. from X to X' and from X to X'' requires to calculate two welfare ratios and should not be carried out by calculating $\mathrm{WR}(X', X'')$. To get an intuition why this is the case, consider $X'' = (20, 0)$. Thus, naively computing $\mathrm{WR}(X', X'')$ yields a ratio equal to unity, since all losses are compensated by equivalent gains. Such an evaluation ignores the change process itself, i.e. it contradicts $\mathrm{WR}(X, X'') \approx 1.1 < 1.3 \approx \mathrm{WR}(X, X')$. For logical reasons, moving from X to X' results in a higher WR, since the relative changes in utility levels is more attractive than in the case of the reform X to X''.

The welfare ratio ratio, as the ratio of welfare gains and welfare losses, is easier to interpret than the net change of SW, either in absolute or in relative terms. In addition, it is interesting to consider whether any attractiveness is due to a few individuals with very large changes. This can be achieved by looking at the modification that is implied by excluding the tails of the change distribution. Given the situation in Definition 1, we introduce the following Notation:

$$\mathrm{SWR}^+_\gamma(X, X') := \max\left\{\frac{\sum_{i\in I^*}(\Delta V_i)^+}{\sum_{i\in I^*}(\Delta V_i)^-} : I^* \subset I \wedge \frac{|I^*|}{|I|} \geq (1-\gamma)\right\} \tag{3}$$

$$\mathrm{SWR}^-_\gamma(X, X') := \min\left\{\frac{\sum_{i\in I^*}(\Delta V_i)^+}{\sum_{i\in I^*}(\Delta V_i)^-} : I^* \subset I \wedge \frac{|I^*|}{|I|} \geq (1-\gamma)\right\}, \tag{4}$$

with $0 \leq \gamma < 1$ and $\wedge$ denoting the logical conjunction. Hence, $\mathrm{SWR}^-_\gamma(X, X')$ quantifies the (substantial) attractiveness that remains when we exclude the most extreme winners. We call $\mathrm{SWR}^-_\gamma(X, X')$ the *substantial welfare ratio*. To obtain a deeper understanding of the dependency of the welfare ratio on the upper and lower tail of the change, a so-called γ-*diagram* visualizes this relationship.

Definition 2 (γ-diagram): We refer to a figure as a γ-*diagram*, when it depicts SWR^+_γ and SWR^-_γ as functions of γ.

Having established the (substantial) welfare ratio, a few remarks on the properties are in order. Many standard utility functions imply a value function that is concave and monotonically increasing in the state variables. If additionally, value functions are the same over the entire population, the welfare ratio cp. prefers policies that lead to more equality in the sense of most inequality measures. In particular, the welfare ratio is consistent with several inequality measure properties. Below, we assume that there is a representative concave and monotonically increasing value function V s.t. $V = V_i$ f.a. i. The welfare ratio then meets the subsequent requirements.

Lemma 1 ***(Monotonicity)***[3]***:***
Let X and X' be two state matrices , with $X_{i,\cdot} \leq X'_{i,\cdot}$ f.a. $i \in I$ and $\exists i \in I : X_{i,\cdot} < X'_{i,\cdot}$. Then

$$WR(X, X') = \infty$$

holds.

Lemma 2 ***(Weak principle of transfers):***
Let $X_{i,k} - \delta > X_{j,k} + \delta > \delta > 0$ for some i, j, k, with $V(X_{i,\cdot} - e_{i,k} \cdot \delta) > V(X_{j,\cdot} + e_{j,k} \cdot \delta)$ then

$$WR(X, X - e_{i,k} \cdot \delta + e_{j,k} \cdot \delta) \geq 1$$

where $e_{l,k}$ denotes a zero-matrix, where only the element in the l'th row and k'th column equals one.

Lemma 3 ***(Consistency with first and second-order dominance)***[4]***:***
Let X and X' be one dimensional and X' dominates X either first or second order. Then

$$WR(X, X') \geq 1.$$

Lemma 4 ***(Principle of population)***[5]***:***
Let X and X' be two state matrices. Define $X^n := X \bigotimes \mathbf{1}_n$ and X'^n accordingly. Then

$$WR(X, X') = WR(X^n, X'^n)$$

for all $n \in \mathbb{N}$.

[3] This property even holds for welfare functions which are not the same for all individuals.
[4] This is also a direct consequence of the properties derived by Shorrocks (1983).
[5] In this definition, $\bigotimes$ denotes the Kronecker product.

For general value functions, the welfare ratio depends on the scale of X. Nevertheless, if we want to drop this property, a welfare function defined as the sum of identical homogeneous value functions implies scale independence.

In conclusion, the substantial welfare ratio is an SW-based tool for analysing (potential) changes in the state variable, e.g. due to a tax reform or a policy decision. It allows for a comparison of different groups (e.g. countries or occupation), and the γ-diagram implements an inherent robustness check. In the next section, we give a sample application.

We close this section with a lemma that states the asymptotic distribution of the welfare ratio and can be used to calculate standard errors, if the welfare ratio of a subsample is interpreted as an estimator for the welfare ratio of the overall population.

Lemma 5 ***(Asymptotic distribution of the welfare ratio):***
Let $(X_N, X'_N), N \in \mathbb{N}$ be a sample of i.i.d. $2K$-dimensional random vectors drawn from (X, X'), representing the state variables of an infinite population. Let V be the value function for all individuals, such that the second moment of $(V(X))^+$, $(V(X))^-$, $(V(X'))^+$, $(V(X'))^-$ exists.[6] *Then*

$$WR(X_N, X'_N) \overset{asymp.}{\sim} GRD,$$

where GRD denotes a Gaussian Ratio distribution.[7]

3 Sample Application

In this section, we use the welfare ratio to analyse two hypothetical tax-reforms in four European countries. We derive a welfare function, based on a simplified life-cycle model. We then use that welfare function in individual wealth and income to analyse a reform that represents an intensification of a progressive taxation and a second reform introducing a wealth tax, for which the revenues are equally redistributed. The data is from the Eurosystem Household Finance and Consumption Survey.[8] We take the variable "*employee income*" as $X_{\cdot,1} =: Y$ and "*net wealth*" as $X_{\cdot,2} =: W$. Finally, we use the difference between "*at what age to retire*" and "*age*" as the number of years that the individual will remain employed.

3.1 Simplified Life-Cycle Model

Consider individuals $i \in I = \{1, ..., N\}$, choosing the stream of consumption C_i^t over a life-cycle from time $t = 0$ to $t = T$, in order to maximize the sum

[6] Note that the existence of the second moment of the state variable itself is not implied. For example, a Pareto distribution with a parameter smaller than 1.5, and a logarithmic value function, are both possible.

[7] Different representations for that distribution function can be found in Marsaglia (1965).

[8] More information on the data can be found in ECB (2013).

of intratemporal utilities $U_i(\cdot)$ discounted by the time preference parameter β. Given an exogenous, constant risk-free rate r_f and an income growth rate g, individuals endowed with initial wealth W_i^0 and initial income level Y_i^0 solve the optimal control problem

$$\max_{C_i^t} \sum_{t=0}^{T} \beta^t U_i(C_i^t). \tag{5}$$

Each individual chooses consumption C_t^i and wealth (i.e. risk-free asset holdings) W_i^{t+1} based on the state $X_i^t := (Y_i^t, W_i^t)$, subject to the budget constraint

$$C_i^t = W_i^t + Y_i^t - \frac{W_i^{t+1}}{1 + r_f}$$

and the evolution of income

$$Y_i^t = Y_i^0 (1 + g)^t,$$

which abstracts from income risks. Following Huggett (1996), we allow individuals to borrow up to a to be specified credit limit $\bar{W}_i^t \leq W_i^t$ and demand $W_i^{T+1} \geq 0$. Thus, we explicitly model the indebtedness present in the data, while maintaining feasibility. The simplifying assumptions of our model are chosen in order to present a minimal working example for the calculation of the SWR. In particular, we consciously refrain from including changing investing opportunities, labour income risks or retirement decisions. However, the concept of the SWR can straightforwardly be applied to models with uncertainty and such additional non-income-related state variables as health, leisure or happiness.

The recursive presentation

$$V_i(X_i^t) = \max_{C_i^t} \left\{ U_i(C_i^t) + \beta V_i(X_i^{t+1}) \right\},$$

enables us to solve the model for any specifications of utility functions $U_i(\cdot)$ by iterating backwards to $t = 0$, in accordance with Samuelson (1969). The resulting value functions $V_i(X_i^0)$ serve as the starting point for calculating the substantial welfare ratio. Note that for any reasonable choices of U_i, the value function has positive first derivatives and negative second derivatives with respect to wealth and income. Given that we are not interested in the evolution of the value function over time, we drop the time indices t, define $V_i(X_i) := V_i(X_i^0)$ and compute the value functions for all $i \in I$, such that we obtain

$$\mathrm{SW}(X) := \sum_{i \in I} V_i(X_i).$$

For the application, we assume a homogeneous population of log-utility individuals with various individual years of working time left. We set $\beta = 0.95$, the interest rate r_f to 0.02 and set the credit limit to $\bar{W}_i^t = -100\ 000$ Euro for each individual i.

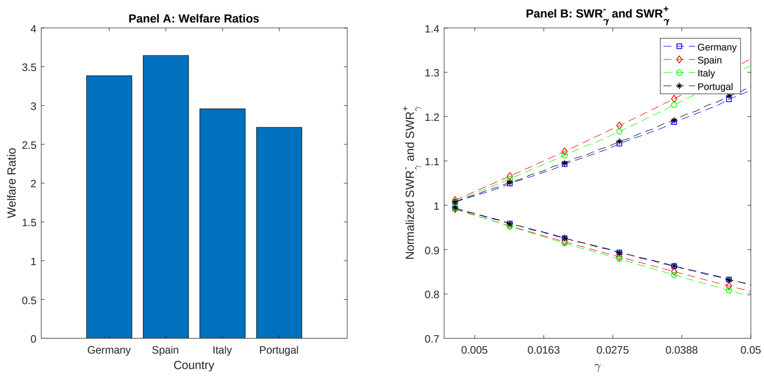

Fig. 1 Intensification of Tax Progression. This figure shows the impact of an intensification of tax progression for different countries. Panel A gives the absolute welfare ratios and Panel B shows the normalized γ diagram.

3.2 Intensification of Tax Progression

The following stylized reform demonstrates the implications of a revenue-neutral intensification of the tax progression. We use gross income and net wealth as the baseline.[9] The tax-change can be described as follows. The lowest 30%$-$quantile receives an income increase of 10%, the next 20% an increase of 5%, from the median to the 70% quantile, the income is reduced by 1% and the 30% largest incomes are reduced by p%, where p is chosen such that the reform is revenue neutral.

In Figure 1 Panel A, the welfare ratio of the described reform is shown. Such a change is considered attractive for all countries. This result is intuitively correct, since the reform implies an increase in income equality.

The corresponding $\gamma-$diagrams in Figure 1 Panel B shed light on the source of the attractiveness. The upper line shows that the value-loss of payers is equally distributed over the top payers. The SWR^--lines show, for all countries, an equally distributed value gain of the reform-winners at the left tail.

3.3 Wealth Taxation

The second hypothetical tax reform represents a wealth tax. Each individual pays 0.01% of his/her wealth and the charged amount is evenly redistributed over all individuals.

In Figure 2 Panel A, the welfare ratio of the wealth-redistribution is shown. The attractiveness of this reform varies strongly between countries, but is attractive for each. In particular for Spain the reform implies a large welfare ratio. This is due to the fact that in the given data set Spain has the most wealthy

[9] A more realistic approach would consider the tax structure in the different countries. We ignore that for the sake of simplicity.

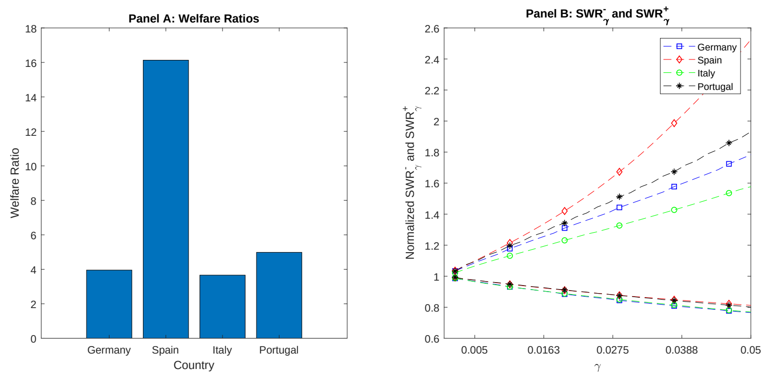

Fig. 2 Wealth Taxation. This figure shows the impact of a wealth taxation for different countries. Panel A gives the absolute welfare ratios and Panel B shows the normalized γ diagram.

individuals and wealth is most unequally distributed. I.e. the reform leads to higher value gains for more individuals, whereas value loss for the net-payers stays low due to the concavity of the value function. The result is intuitively correct, since the reform implies an increase in wealth equality. Note that the reform is neutral concerning tax revenue.

The corresponding $\gamma-$diagrams for the 5%-interval in Figure 2 Panel B describe the distribution of value gains and losses. The SWR^--line is very stable and decreases only slightly for all countries, indicating equal gains on the left tail of the value distribution. In contrast, we see a strong increase in SWR^+ which implies that the value loss occurs at the right tail. The effect is most evident for Spain. Here the advantage of the SWR-diagram becomes evident, as it identifies the extreme structure of the right tail of the wealth distribution in Spain in the given data set. I.e. the absolute values and the inequality at the right tail is by far higher in comparison to the other three countries.

3.4 Comparison of Two Reform Outcomes

In Section 2 we present two reforms with equivalent final states and show that even though the resulting social welfare is equal across both reforms, the way of reallocation matters for evaluation. In contrast, in a case where the two reforms are exclusive alternatives, we are more interested in comparing the final states of the reforms than the process of change. In this case, the resulting WRs of both reforms cannot be used for comparison, since the WR as a relative measure is not transitive. Instead a direct comparison by means of a single WR of both final states allows to assess the differences in SW implied by both reform outcomes.

For the sample tax reforms, the WR labels both as advantageous and a policy maker who only cares about social welfare maximisation should carry out both of them. In order to answer the question: "*How advantageous is reform*

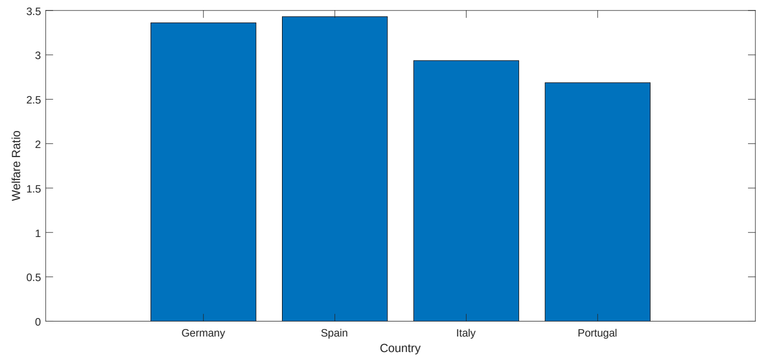

Fig. 3 Comparison of Reform Outcomes. This figure shows the absolute welfare ratios for comparing reform outcomes 3.2. and 3.3. for different countries.

outcome 3.2 compared to reform outcome 3.3?" we calculate the WR of the two final states. Figure 3 depicts the corresponding welfare ratios.

For all countries the intensification of tax progression, affecting the whole lifetime, is more attractive than an one-time wealth taxation. This example particularly shows, that, given mutually exclusive alternatives, the absolute changes in the state variables are of interest. Nonetheless, the welfare ratio allows to express the resulting attractiveness in the intuitive language of a relative measure.

4 Conclusion

This paper introduces new tools, namely the welfare ratio, the substantial welfare ratio and the γ-diagram, in order to analyse the attractiveness of a policy decision in terms of a change in social welfare. The proposed tools are theoretically sound, incorporate a robustness check and fulfil a variety of desirable properties, known from the literature on inequality and performance measurement. An application to income and wealth data for four European countries shows the heterogeneous attractiveness of different tax-reforms and ease of application.

A topic of further research might be the investigation of the welfare ratio for alternative forms of the welfare function, and the empirical application to specific real-world policy decisions.

References

Alvaredo, F. and E. Saez (2009). Income and wealth concentration in spain from a historical and fiscal perspective. *Journal of the European Economic Association 7*(5), 1140–1167.

Aronsson, T. (2010). Welfare measurement, involuntary unemployment, and heterogeneity. *Review of Income and Wealth 56*(3), 559–571.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory 2*, 244–263.

Azpitarte, F. (2012). Measuring poverty using both income and wealth: a cross-country comparison between the u.s. and spain. *Review of Income and Wealth 58*(1), 24–50.

Bergson, A. (1938). A reformulation of certain aspects of welfare economics. *The Quarterly Journal of Economics*, 310–334.

Bernardo, A. E. and O. Ledoit (2000). Gain, loss, and asset pricing. *Journal of Political Economy 108*(1), 144–172.

Bradley, H. (2015). *Fractured identities: Changing patterns of inequality.* John Wiley & Sons.

Brandolini, A., S. Magri, and T. M. Smeeding (2010). Asset-based measurement of poverty. *Journal of Policy Analysis and Management, 29*(2), 267–284.

Castañeda, A., J. Díaz-Gimíenez, and J.-V. Ríos-Rull (2003). Accounting for the u.s. earnings and wealth inequality. *Journal of Political Economy 111*(4), 818–857.

Chauvel, L. (2016). The intensity and shape of inequality: the abg method of distributional analysis. *Review of Income and Wealth 62*(1), 52–68.

Cowell, F. (2011). *Measuring inequality.* OUP Oxford.

Dalton, H. (1920). The measurement of the inequality of incomes. *The Economic Journal 30*(119), 348–361.

De Langhe, B. and S. Puntoni (2015). Bang for the buck: gain-loss ratio as a driver of judgment and choice. *Management Science 61*(5), 1137–1163.

Dorling, D. (2015). *Injustice: Why social inequality still persists.* Policy Press.

Dutta, I. and J. Foster (2013). Inequality of happiness in the us: 1972–2010. *Review of Income and Wealth 59*(3), 393–415.

ECB (2013). The eurosystem household finance and consumption survey-results from the first wave. Technical report, European Central Bank.

Fleurbaey, M. (2009). The quest for a measure of social welfare. *Journal of Economic Literature 47*(4), 1029–1075.

Fleurbaey, M. (2015). Beyond income and wealth. *Review of Income and Wealth 61*(2), 199–219.

Grodner, A. and T. J. Kniesner (2008). Distribution of wealth and interdependent preferences. *IZA Discussion Paper No. 3684.*

Hadar, J. and W. R. Russell (1969). Rules for ordering uncertain prospects. *The American Economic Review 59*(1), 25–34.

Headey, B. and M. Wooden (2004). The effects of wealth and income on subjective well-being and ill-being. *The Economic Record 80*(Special Issue), S24–S33.

Hotchkiss, J. L., R. E. Moore, and F. Rios-Avila (2012). Assessing the welfare impact of tax reform: a case study of the 2001 u.s. tax cut. *Review of Income and Wealth 58*(2), 233–256.

Huggett, M. (1996). Wealth distributions in life-cycle economies. *Journal of Monetary Economics 38*, 469–494.

Krusell, P. and A. A. Smith (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy 106*(5), 867–896.

Kuznets, S. (1955). Economic growth and economic inequality. *The American Economic Review XLV*(1), 1–28.

Marsaglia, G. (1965). Ratios of normal variables and ratios of sums of uniform variables. *Journal of the American Statistical Association 60*(309), 193–204.

Ng, Y.-K. (1983). *Welfare economics*. Springer.

Ng, Y.-K. (2003). From preference to happiness: Towards a more complete welfare economics. *Social Choice and Welfare 20*(2), 307–350.

Peichl, A., T. Schaefer, and C. Scheicher (2010). Measuring richness and poverty: a micro data application to europe and germany. *Review of Income and Wealth 56*(3), 597–619.

Piketty, T. and E. Saez (2003). Income inequality in the united states, 1913-1998. *The Quarterly Journal of Economics 118*(1), 1–39.

Piketty, T. and G. Zucman (2014). Capital is back: Wealth-income ratios in rich countries 1700-2010. *The Quarterly Journal of Economics 129*, 1255–1310.

Quadrini, V. and J.-V. Ríos-Rull (1997). Understanding the u.s. distribution of wealth. *Federal Reserve Bank of Minneapolis Quarterly Review 21*(2), 22–36.

Samuelson, P. A. (1948). Foundations of economic analysis.

Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics 51*(3), 239–246.

Shorrocks, A. F. (1983). Ranking income distributions. *Economica 50*(197), 3–17.

Stiglitz, J. (2012). *The price of inequality*. Penguin UK.

Voelzke, J. (2015). Weakening the gain–loss-ratio measure to make it stronger. *Finance Research Letters 12*, 58–66.

Weisbrod, B. A. and W. L. Hansen (1968). An income-net worth approach to measuring economic welfare. *The American Economic Review 58*(5), 1315–1329.

Appendix

In the appendix, we provide proofs for the properties stated in Section 2. In the following analysis, we assume that V is a concave and monotonically increasing value function V s.t $V = V_i$ f.a. $i \in I$. We deal with the properties in the order of their occurrence in the text.

Lemma 1 ***(Monotonicity):***
Let X and X' be two state matrices , with $X_{i,\cdot} \leq X'_{i,\cdot}$ f.a. $i \in I$ and $\exists i \in I : X_{i,\cdot} < X'_{i,\cdot}$. Then

$$WR(X, X') = \infty$$

holds.

Proof. $\exists i \in I : X_{i,\cdot} < X'_{i,\cdot}$ implies $\sum_{i \in I}(\Delta V_i)^+ > 0$ and $X_{i,\cdot} \geq X'_{i,\cdot}$ implies $\sum_{i \in I}(\Delta V_i)^- = 0$. Hence, $\mathrm{WR}(X, X') = \infty$ per definition (analytic continuation). □

Lemma 2 ***(Weak principle of transfers):***
Let $X_{i,k} - \delta > X_{j,k} + \delta > \delta > 0$ for some i, j, k, with $V(X_{i,\cdot} - e_{i,k} \cdot \delta) > V(X_{j,\cdot} + e_{j,k} \cdot \delta)$ then

$$WR(X, X - e_{i,k} \cdot \delta + e_{j,k} \cdot \delta) \geq 1$$

where $e_{l,k}$ denotes a zero-matrix, where only the element in the l'th row and k'th column equals one.

Proof. Direct implication of the fact that V is concave and non-decreasing:

$$\begin{aligned} \mathrm{WR}(X, X - e_{i,k} \cdot \delta + e_{j,k} \cdot \delta) &= \frac{\sum_{l \in I}(\Delta V_l)^+}{\sum_{l \in I}(\Delta V_l)^-} \\ &\overset{nondec.}{=} \frac{\Delta V_j}{\Delta V_i} \\ &\overset{concave}{\geq} 1 \end{aligned}$$

□

Lemma 3 ***(Consistency with first and second-order dominance):***
Let X and X' be one dimensional and X' dominates X either first or second order. Then

$$WR(X, X') \geq 1.$$

Proof. We provide the proof for a continuous population, analogue to Hadar and Russell (1969). The discrete case follows analogous (using the mean value theorem) and is given in the paper of Hadar and Russell (1969) as well. Since first-order dominance implies second-order dominance, we assume w.l.o.g. second order dominance, i.e.
$\int_{\underline{x}}^{b} F_X(x)dx \geq \int_{\underline{x}}^{b} F_{X'}(x)dx$ for all $b \in R$, where F_X and $F_{X'}$ denotes the cumulative distribution of X and X' respectively and further $\underline{x}, \overline{x} \in \mathbb{R}$ be a strict left respective right limit of X and X'. Applying integration by part twice (marked with *p.i.*) and using $V'(x) \geq 0$ and $V''(x) \leq 0$ leads to:

$$\begin{aligned}
&\int V(x)f_X(x)dx - \int V(x)f_{X'}(x)dx \\
=& \int V(x)(f_X(x) - f_{X'}(x))dx \\
\overset{p.i.}{=}& \underbrace{[V(x)(F_X(x) - F_{X'}(x))]_{\underline{x}}^{\overline{x}}}_{=0} \\
&- \int_{\underline{x}}^{\overline{x}} V'(y)(F_X(y) - F_{X'}(y))dy \\
\overset{p.i.}{=}& -[\underbrace{V'(x)}_{\geq 0} \underbrace{\int_{\underline{x}}^{x} (F_X(y) - F_{X'}(y))dy]_{\underline{x}}^{\overline{x}}}_{\geq 0} \\
&+ \int_{\underline{x}}^{\overline{x}} \underbrace{V''(x)}_{\leq 0} \underbrace{\int_{\underline{x}}^{x} (F_X(y) - F_{X'}(y))dy}_{\geq 0} \, dx \leq 0
\end{aligned}$$

Hence, the SW(X') is larger than SW(X) and WR(X, X') ≥ 1 □

Lemma 4 ***(Principle of population):***
Let X and X' be two state matrices. Define $X^n := X \bigotimes \mathbf{1}_n$ and X'^n accordingly . Then

$$WR(X, X') = WR(X^n, X'^n)$$

for all $n \in \mathbb{N}$.

Proof.

$$\begin{aligned}\mathrm{WR}(X,X') &= \frac{\sum_{i\in I}(\Delta V_i)^+}{\sum_{i\in I}(\Delta V_i)^-}\\ &= \frac{n\sum_{i\in I}(\Delta V_i)^+}{n\sum_{i\in I}(\Delta V_i)^-}\\ &= \frac{\sum_{i\in I^*}(\Delta V_i)^+}{\sum_{i\in I^*}(\Delta V_i)^-}\\ &= \mathrm{WR}(X^n, X'^n),\end{aligned}$$

where I^* denotes the unification of the indices of the $n \times |I|$ individuals. □

Lemma 5 ***(Asymptotic distribution of the welfare ratio):***
Let $(X_N, X'_N), N \in \mathbb{N}$ be a sample of i.i.d. $2K$-dimensional random vectors drawn from (X, X'), representing the state variables of an infinite population. Let V be the value function for all individuals, such that the second moment of $(V(X) - V(X'))^+$ and $(V(X) - V(X'))^-$ exists.[10] *Then*

$$WR(X_N, X'_N) \overset{asymp.}{\sim} GRD,$$

where GRD denotes a Gaussian Ratio distribution.[11]

Proof. The multivariate central limit theorem states

$$\frac{\sum_{i\in I_N}(\Delta V_i)^+}{N} \overset{dist.}{\to} \mathcal{N}(E((\Delta V_1)^+), \Sigma_Z),$$

where Σ_Z is the covariance matrix of $Z := (V(X) - V(X'))^+$. Analogous results hold for the corresponding negative parts $(\cdot)^-$. Hence,

$$\begin{aligned}&\frac{\sum_{i\in I_N}(\Delta V_i)^+}{\sum_{i\in I_N}(\Delta V_i)^-}\\ =&\frac{\frac{\sum_{i\in I_N}(\Delta V_i)^+}{N}}{\frac{\sum_{i\in I_N}(\Delta V_i)^-}{N}} \overset{dist.}{\to} GRD.\end{aligned}$$

□

[10] Note that the existence of the second moment of the state variable itself is not implied. For example, both a Pareto distribution with a parameter smaller than 1.5 and a logarithmic value function are possible.

[11] Different representations for this distribution function can be found in Marsaglia (1965).