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Exact expectations – Efficient calculation of DSGE models

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Abstract

Global solution methods for *dynamic stochastic general equilibrium* (DSGE) models are acurrate but computationally expensive. In particular computing conditional expectations for numerous points in the state-space leads to significant complexity. In the present paper, I show how to remove the majority of calculations required for the evaluation of conditional expectations. Therefore I replace the approximated conditional expectation obtained by e.g. quadrature rules with an exact expectation. Further, similar to Judd et al. (2011), the required integrals are evaluated at the initial stage of the algorithm. I adopt Chebyshev polynomials as basis functions and provide a general framework. Subsequently, I adapt the technique to the neoclassical model with recursive utility and labor choice.

Keywords: Projection, Precomputation, DSGE, Complexity reduction
JEL: C63, C68

1. Introduction

Global solution methods for *dynamic stochastic general equilibrium* (DSGE) models are computationally expensive and hard to implement. In contrast, local perturbation solutions are easily implemented and require minor computational effort. Consequently, estimations and policy analysis, which require repeated solutions of the model, favor local methods due to their superiority in computational time. But as Fernández-Villaverde and Levintal (2016), Levintal (2016) and Branger et al. (2015) argue, the non-linearities in modern models, induced by disasters, long-run-risks, adjustment costs, high-volatilities etc., lead to significant inaccuracies for local perturbations. Whereas this carries over even to high-order perturbations, global solutions are not prone to these errors. Consequently, as the accuracy of quantitative models is of highest importance, accelerating global solution methods is a necessary field of research. Therefore note that a solution method can be accelerated in two dimensions. On the one hand, *technological acceleration*, e.g. parallel computing on GPU/CPU as in Aldrich et al. (2011), or using more machine-oriented programming languages accelerates computations. On the other hand, *complexity reduction* can significantly decrease computing time. The present paper aims at the latter.
Complexity in the global solution of DSGE models arises in particular from the *general equilibrium* and the occurring *stochastics*. Firstly, a *general equilibrium* requires the solution to satisfy a system of stochastic difference equations globally over the state-space. As an exact solution is generally unknown, a collocation type method solves the system of equations at a particular set of points of the state-space. Still, the required size of this set grows rapidly with growing number of state variables. The resulting computational effort can be reduced by using sparse grids, e.g. a Smolyak (1963) grid and corresponding approximations as in Judd et al. (2014). Secondly, the *stochastic* components of the model require the calculation of (multiple) conditional expectations at each point of the grid. Commonly these expected values are approximated using Gauss-Hermite quadrature rules or a Tauchen (1986) approximation. Both of these techniques require the evaluation of an integrand at several points in the state-space, constituting a bottleneck for the computation. In the present paper, I show how to remove this limitation by replacing the approximated expectation

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with an exact expectation over an approximated integrand, which is evaluated at the initial stage of the algorithm. In this regard, the precomputation technique developed in Judd et al. (2011) is the point of departure for the present paper. I adopt Chebyshev polynomials as basis functions and provide a general framework. Subsequently, I adapt the technique to the neoclassical model with recursive utility and labor choice.
The rest of the paper is as follows: Section 2 establishes the technique and provides a simple example. Section 3 applies the developed method technique to the benchmark model. Subsequently to a short discussion of the technique in Section 4, Section 5 concludes.

2. Technique

2.1. Notation

Let $\mathcal{T}(x)$ be a vector of functions, which stacks unidimensional Chebyshev basis functions of degree $j \in \{2, ..., N\}$ with

$$\mathcal{T}_j(x) = 2xT_{j-1}(x) - T_{j-2}(x),$$

$\mathcal{T}_0(x) = 1$ and $\mathcal{T}_1(x) = x$. Additionally, define the rescaling function

$$\varphi(x) = 2\frac{x - \underline{x}}{\overline{x} - \underline{x}} - 1,$$

which maps values x from $[\underline{x}, \overline{x}]$ into the domain of the Chebyshev polynomials, $[-1, 1]$. Note that for a state variable $s_{i,t} = s_{i,t,A} + s_{i,t,B}$, we have

$$\begin{aligned}\varphi_i(s_{i,A,t} + s_{i,B,t}) &= 2\frac{(s_{i,A,t} + s_{i,B,t}) - \underline{s}_i}{\overline{s}_i - \underline{s}_i} - 1 \\ &= \underbrace{2\frac{s_{i,A,t} - \underline{s}_i}{\overline{s}_i - \underline{s}_i} - 1}_{=\varphi_i(s_{i,A,t}) =: \mathcal{A}_{i,t}} + \underbrace{2\frac{s_{i,B,t}}{\overline{s}_i - \underline{s}_i}}_{=: \mathcal{B}_{i,t}},\end{aligned} \tag{1}$$

for arbitrary $s_{i,t}$. Further, defining a coefficient vector $\mathbf{b} \in \mathbb{R}^{|I|}$, the polynomial function

$$\begin{aligned}\Psi(\mathbf{s}_t, \mathbf{b}, I) &= \left(\bigotimes_{i=1}^{d} \mathcal{T}(\varphi_i(s_{i,t}))\right)_I \mathbf{b} \\ &= \Phi_I(\mathbf{s}_t)\mathbf{b}\end{aligned}$$

is the linear combination of products of basis functions in d state variables $s_{i,t}$. The indexing set I allows choosing specific products of basis functions, depending on the choice of the constructed polynomial. For example the tensor product of Chebyshev basis functions with maximal degree N in d dimensions requires $I = \{1, 2, \ldots, (N+1)^d\}$, whereas a complete polynomial or a Smolyak (1963) type approximation allows for smaller sets.

Denoting predetermined/purely endogenous state variables $\mathbf{s}_t^{en}$ and (partly) exogenous state variables $\mathbf{s}_t^{ex}$, the complete state vector can be partitioned to $\mathbf{s}_t = (\mathbf{s}_t^{en}, \mathbf{s}_t^{ex})$, where $\mathbf{s}_t^{en} = (s_{1,t}, ..., s_{d_{en},t})$ and $\mathbf{s}_t^{ex} = (s_{d_{en}+1,t}, ..., s_{d,t})$. Here predetermination of a variable x_{t+1} is understood as t measurability. By construction the product of Chebyshev basis functions of degrees $\mathbf{j} = (j_1, ..., j_d)$ is the k-th element in vector $\Phi_I(\mathbf{s}_t)$,

$$\Phi_k(\mathbf{s}_t) = \prod_{i=1}^{d_{en}} \mathcal{T}_{j_i}(\varphi_i(s_{i,t})) \prod_{i=1+d_{en}}^{d} \mathcal{T}_{j_i}(\varphi_i(s_{i,t})),$$

such that

$$k = 1 + \sum_{i=1}^{d} (N+1)^{d-i} j_i.$$

2.2. Precomputation

Given the notation, the conditional expectation

$$\mathrm{E}_t \Psi(\mathbf{s}_{t+1}, \mathbf{b}, I), \tag{2}$$

is the weighted sum of conditional expectations of $\Phi_k(\mathbf{s}_t)$ with $k \in I$. Thus calculating (2) can be partitioned into calculating

$$\mathrm{E}_t \Phi_k(\mathbf{s}_{t+1}) = \prod_{i=1}^{d_{en}} \mathcal{T}_{j_i}(\varphi_i(s_{i,t+1})) \mathrm{E}_t \prod_{i=1+d_{en}}^{d} \mathcal{T}_{j_i}(\varphi_i(s_{i,t+1})) \tag{3}$$

for all $k \in I$ and subsequently taking the $\mathbf{b}$ weighted sum. For the proceeding steps, I make the following assumptions, which are close to the assumptions in Judd et al. (2011):

Assumption 1. *Exogenous state variables $\mathbf{s}_t^{ex}$ are sums of a predetermined components $\mathbf{s}_{A,t}^{ex}$ and a stochastic components $\mathbf{s}_{B,t}^{ex}$.*

Assumption 2. *All stochastic components $\mathbf{s}_{B,t}^{ex}$ are mutually independent and iid over time.*

Assumption 3. *Moments of $\mathbf{s}_{B,t}^{ex}$ up to order N exist and are finite.*

Section 4 further discusses the implications and the applicability of the assumptions.
Under Assumptions 1 and 2, using $\mathcal{A}_{i,t+1}$ and $\mathcal{B}_{i,t+1}$ as defined in (1), the second term in equation (3) can be rewritten as

$$\begin{aligned}
\mathrm{E}_t \prod_{i=1+d_{en}}^{d} \mathcal{T}_{j_i}(\varphi_i(s_{i,t+1})) &= \mathrm{E}_t \prod_{i=1+d_{en}}^{d} \mathcal{T}_{j_i}(\mathcal{A}_{i,t+1} + \mathcal{B}_{i,t+1}) \\
&= \mathrm{E}_t \prod_{i=1+d_{en}}^{d} \left(\mathcal{T}_{j_i}(\mathcal{A}_{i,t+1}) + \sum_{n=1}^{j_i} a_n f_n(\mathcal{A}_{i,t+1}, \mathcal{B}_{i,t+1}) \right) \\
&= \prod_{i=1+d_{en}}^{d} \left(\mathcal{T}_{j_i}(\mathcal{A}_{i,t+1}) + \sum_{n=1}^{j_i} a_n \mathrm{E}_t f_n(\mathcal{A}_{i,t+1}, \mathcal{B}_{i,t+1}) \right)
\end{aligned} \tag{4}$$

where

$$f_n(\mathcal{A}_{i,t+1}, \mathcal{B}_{i,t+1}) := \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{A}_{i,t+1}^k \mathcal{B}_{i,t+1}^{n-k}$$

and a_n are the coefficients in front of power n terms in $\mathcal{T}_{j_i}$. Assumptions 2 and 3 allow to rewrite $\mathrm{E}_t f_n(\mathcal{A}_{i,t+1}, \mathcal{B}_{i,t+1})$ as $f_n^*(\mathcal{A}_{i,t+1})$ and yield

$$\begin{aligned}
\mathrm{E}_t \prod_{i=1+d_{en}}^{d} \mathcal{T}_{j_i}(\varphi_i(s_{i,t+1})) &= \prod_{i=1+d_{en}}^{d} \left(\mathcal{T}_{j_i}(\mathcal{A}_{i,t+1}) + \sum_{n=1}^{j_i} a_n f_n^*(\mathcal{A}_{i,t+1}) \right) \\
&= \prod_{i=1+d_{en}}^{d} \mathcal{T}_{j_i}(\varphi_i(s_{i,A,t+1})) + \mathcal{C}_{j_i}^*(\varphi_i(s_{i,A,t+1})),
\end{aligned}$$

where $\mathcal{C}_{j_i}^*$ contains all cross products and $\mathbf{s}_{A,t} := (\mathbf{s}_t^{en}, s_{d_{en}+1,A,t}, ..., s_{d,A,t})$. Consequently, the conditional expectation (3) can be rewritten as

$$\mathrm{E}_t \Phi_k(\mathbf{s}_{t+1}) = \Phi_k(\mathbf{s}_{i,A,t+1}) + \Theta_k(\mathbf{s}_{i,A,t+1}), \tag{5}$$

where

$$\Theta_k(\mathbf{s}_{i,A,t+1}) = \prod_{i=1}^{d_{en}} \mathcal{T}_{j_i}(\varphi_i(s_{i,t+1}))\mathcal{C}^*_{j_i}(\varphi_i(s_{i,A,t+1})).$$

Finally the conditional expectation in equation (2) is

$$\mathrm{E}_t\Psi(\mathbf{s}_{t+1}, \mathbf{b}, I) = \underbrace{\Psi(\mathbf{s}_{A,t+1}, \mathbf{b}, I)}_{:=\Phi_I(\mathbf{s}_{A,t+1})\mathbf{b}} + \underbrace{\Upsilon(\mathbf{s}_{A,t+1}, \mathbf{b}, I)}_{:=\Theta_I(\mathbf{s}_{A,t+1})\mathbf{b}}$$

where Υ captures all uncertainty.

2.3. Example

For illustration, I consider the stylized example of one endogenous and one exogenous state variable. Given this setting,

$$\Phi_k(\mathbf{s}_t) = \mathcal{T}_{j_1}(\varphi_1(s_{1,t}))\mathcal{T}_{j_2}(\varphi(s_{2,t}))$$

and

$$\mathrm{E}_t\Phi_k(\mathbf{s}_{t+1}) = \mathcal{T}_{j_1}(\varphi_1(s_{1,t+1}))\mathrm{E}_t\mathcal{T}_{j_2}(\varphi_2(s_{2,t+1})). \tag{6}$$

Now, in order to follow up (4), consider the Chebyshev basis function of degree 2 and the corresponding conditional expectation

$$\begin{aligned} \mathcal{T}_2(x) &= 2x^2 - 1, \\ \mathrm{E}_t\mathcal{T}_2(\mathcal{A}_{t+1} + \mathcal{B}_{t+1}) &= 2\mathrm{E}_t[(\mathcal{A}_{t+1} + \mathcal{B}_{t+1})^2] - 1, \end{aligned}$$

with

$$\begin{aligned} \mathrm{E}_t\left[(\mathcal{A}_{t+1} + \mathcal{B}_{t+1})^2\right] &= \mathrm{E}_t[(\mathcal{A}^2_{t+1} + 2\mathcal{A}_{t+1}\mathcal{B}_{t+1} + \mathcal{B}^2_{t+1})] \\ &= \mathcal{A}^2_{t+1} + 2\mathcal{A}_{t+1}\mathrm{E}_t\mathcal{B}_{t+1} + \mathrm{E}_t\mathcal{B}^2_{t+1}. \end{aligned}$$

Substitution and rearranging yields

$$\begin{aligned} \mathrm{E}_t\mathcal{T}_2(\mathcal{A}_{t+1} + \mathcal{B}_{t+1}) &= 2\mathcal{A}^2_{t+1} - 1 + 2(2\mathcal{A}_{t+1}\mathrm{E}_t\mathcal{B}_{t+1} + \mathrm{E}_t\mathcal{B}^2_{t+1}) \\ &= \mathcal{T}_2(\mathcal{A}_{t+1}) + 2f^*_2(\mathcal{A}_{t+1}), \end{aligned}$$

where $f^*_2(\mathcal{A}_{t+1})$ is determined by the exponents in $\mathcal{T}_2$. The binomial theorem yields the general form of functions f^*_n as

$$f^*_n(\mathcal{A}_{t+1}) = \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{A}^k_{t+1}\mathrm{E}_t\mathcal{B}^{n-k}_{t+1},$$

which allows to calculate the conditional expectation over Chebyshev polynomials of degree n as

$$\mathrm{E}_t\mathcal{T}_j(\mathcal{A}_{t+1} + \mathcal{B}_{t+1}) = \mathcal{T}_j(\mathcal{A}_{t+1}) + \sum_{n=1}^{j} a_n f^*_n(\mathcal{A}_{t+1}),$$

given the existence of all occuring moments of $\mathcal{B}_{t+1}$. Using this result in (6) yields

$$\mathrm{E}_t\Phi_k(\mathbf{s}_{t+1}) = \Phi_k(\mathbf{s}_{A,t+1}) + \Theta_k(\mathbf{s}_{A,t+1}), \tag{7}$$

with $\mathbf{s}_{A,t+1} = (s^{en}_{t+1}, s^{ex}_{A,t+1})$.

3. Application

3.1. Model

The present section applies the developed method to the stochastic neoclassical model with recursive preferences and labor choice. The representative agent maximizes her utility over the consumption bundle $\tilde{c}_t$, consisting of consumption c_t and leisure $1 - l_t$. Production is of the Cobb-Douglas type with capital stock k_t, labor l_t and an AR(1) specification of log-productivity z_t. Prior to applying the precomputation technique, the well-known system of equations has to be transformed as follows. The value function is

$$V_t = \left((1-\beta)\tilde{c}_t^{1-1/\psi} + \beta \left(E_t V_{t+1}^{1-\gamma} \right)^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}},$$

where defining

$$\mathcal{P}_t := V_t^{1-\gamma}$$

yields the transformed value function as

$$\mathcal{P}_t = \left((1-\beta)\tilde{c}_t^{1-1/\psi} + \beta (\mathrm{E}_t \mathcal{P}_{t+1})^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1-\gamma}{1-1/\psi}}. \tag{8}$$

The corresponding Euler equation

$$c_t^{-1}\tilde{c}_t^{1-1/\psi}(\mathrm{E}_t V_{t+1}^{1-\gamma})^{1-1/\theta} = \beta \mathrm{E}_t \left[c_{t+1}^{-1}\tilde{c}_{t+1}^{1-1/\psi}(V_{t+1}^{1-\gamma})^{1-1/\theta} r_{t+1}^k \right]$$

is transformed by defining

$$\mathcal{Q}_t := c_t^{-1}\tilde{c}_t^{1-1/\psi}(V_t^{1-\gamma})^{1-1/\theta} r_t^k,$$

and thus

$$\frac{\mathcal{Q}_t(\mathrm{E}_t \mathcal{P}_{t+1})^{1-1/\theta}}{\mathcal{P}_t^{1-1/\theta} r_t^k} = \beta \mathrm{E}_t \mathcal{Q}_{t+1}. \tag{9}$$

Augmenting the transformed value function and Euler equation by the consumption-labor equilibrium

$$c_t = \frac{\nu}{1-\nu}(1-\alpha)\exp(z_t)k_t^\alpha l_t^{1-\alpha}(1-l_t), \tag{10}$$

the definitions of the consumption bundle and return on capital

$$\tilde{c}_t = c_t^\nu (1-l_t)^{1-\nu}, \tag{11}$$

$$r_t^k = \alpha \exp(z_t) k_t^{\alpha-1} l^{1-\alpha} + (1-\delta), \tag{12}$$

the law of motions for the capital stock and log-productivity

$$k_{t+1} = (1-\delta)k_t + z_t k_t^\alpha l^{1-\alpha} - c_t, \tag{13}$$

$$z_{t+1} = \rho z_t + \epsilon_{t+1} \tag{14}$$

closes the model. Having parameterized the policy for labor $l_t = \Psi(\mathbf{s}_t, \mathbf{b}_l, I)$, the transformed value function $\mathcal{P}_t = \Psi(\mathbf{s}_t, \mathbf{b}_p, I)$ and the integrand $\mathcal{Q}_t = \Psi(\mathbf{s}_t, \mathbf{b}_q, I)$, the coefficients $\mathbf{b}_q$ are uniquely determined by

$$\mathbf{b}_q = \Phi_I(\mathbf{S}_t)^{-1}(\mathbf{c}_t^{-1}\tilde{\mathbf{c}}_t^{1-\frac{1}{\psi}}\mathcal{P}_t^{1-1/\theta}\mathbf{r}_t^k),$$

Here $\mathbf{S}_t = (\mathbf{s}_t^1, ..., \mathbf{s}_t^q)'$ is a matrix which stacks state vectors $\mathbf{s}_t$, evaluated at a set of grid points and remaining bold variables are vectors of variables evaluated at states $\mathbf{S}_t$. Section 4 discusses this approach further.
At last, note that Assumptions 1 to 3 are met with $s_t^{en} = k_t$ and $s_{A,t}^{ex} = \rho a_{t-1}$ with $s_{B,t}^{ex} = \epsilon_t$ and $\epsilon_t \sim \mathcal{N}(0, \sigma)$.

3.2. Solution

The solution to the model is determined by a policy function for leisure and the value function, which jointly satisfy equations (8) and(9). For the transformed model, this requires determination of the coefficient vectors $\mathbf{b}_p$ and $\mathbf{b}_l$. Therefore I adopt a collocation point criterion and construct a rectangular grid using the zeros of the $N+1$ degree Chebyshev polynomials or respectively the Smolyak (1963) zeros. The latter yields a sparse grid and thus by using the respective Smolyak (1963) polynomial a possible *curse of dimensionality* is taken account for.

As the global solution requires solving the system of equations on each node of the grid, the starting values of $\mathbf{b}_p$ and $\mathbf{b}_l$ are of particular importance. I adopt the algorithm developed in Klein (2000), which yields a solution to the linearized model by applying the generalized Schur decomposition. Using the obtained starting values, the follow-up Newton-solver converges to the global solution in few iteration steps. In the following example, the model is parameterized by choosing $\beta = 0.99$, $\delta = 0.025$, $\gamma = 5$, $\psi = 1.5$, $\alpha = 0.33$, $\nu = 0.357$, $\rho = 0.95$, $\sigma =$, while $\theta = (1-\gamma)/(1-(1/\psi))$.

Having solved for the coefficient vectors $\mathbf{b}_p$ and $\mathbf{b}_l$, the corresponding policies for V_t and l_t are recovered and used to simulate the model. Further they allow for the computation of Euler equation errors (EEE) as introduced by Judd (1992). I apply a 10-node Gauss-Hermite quadrature rule and calculate the absolute errors in log10 units. Consequently, a value of -2 is to be interpreted as a 1% error in consumption, a value of -3 as a 0.1% error and so on. Table 1 shows the maximal and mean EEE over the grid for different approximation schemes. Corresponding figures are provided in the Appendix.

Type	max. Degree	max(|EEE|)	mean(|EEE|)
Tensor	3	-1.98	-2.50
Tensor	5	-2.80	-3.70
Tensor	12	-6.88	-7.58
Smolyak	5	-2.69	-3.47
Smolyak	9	-4.02	-5.05

Table 1: Euler equation Errors (EEE) for a 10-node Gauss-Hermite Quadrature rule on a rectangular grid for different Tensor/Smolyak Polynomials

4. Discussion

The present section shortly discusses the implementation and application of the method. The method is adaptable for additional state variables and thus generally applicable. For example adding predetermined labor choice and a long run factor as in Croce (2014), gives $\mathbf{s}_t^{en} = (k_t, l_{t-1})$ and $\mathbf{s}_t^{ex} = (a_t, x_t)$ with $\mathbf{s}_{A,t+1}^{ex} = (\mu + x_t, \rho x_t)$ and $\mathbf{s}_{B,t+1}^{ex} = (\epsilon_{a,t+1}, \epsilon_{x,t+1})$. Equivalently any model meeting the assumptions is easily adopted for suiting definitions of the integrands. Parameterizing the integrand $\mathcal{Q}_t$ is nothing new to the literature and similar to Judd et al. (2011) or the parameterized expectations approach (PEA), see den Haan and Marcet (1990) and Marcet and Lorenzoni (1998). Instead of using a regression step to determine $\mathbf{b}_q$, I adopt Lagrange interpolation as in Judd et al. (2014). These authors also show that by adopting Chebyshev basis functions, the matrix inverse of $\Phi(\mathbf{S}_t)_I$ is well behaved. Apart from parameterizing complete integrands in the Euler equation, conditional expectation over the Value function are readily available, providing a valuable advantage for recursive utilities of the Epstein and Zin (1989) type.

Precomputation constructs the conditional expectation prior to starting any iteration and thus adds a fixed cost in time. Fortunately, the required functions for Θ are by-products of the construction of Φ and straightforward to implement e.g. using MATLAB's symbolic toolbox. In order to assess the complexity reduction, let d be the number of exogenous state variables, e the number of conditional expectations required, p the number of iteration steps and m the number of Gauss-Hermite integration nodes. Assuming that the evaluation of Ψ and Υ are comparable, pem^d evaluations of integrands are required for the quadrature rule and $2pe$ for precomputation method *at each grid point*.

Whereas Assumptions 1-3 vastly simplify necessary calculations, nevertheless they nest the majority of production based asset pricing models. Assumption 1 represents the frequently used AR(1) specification for e.g. log-productivity, where $s^{ex}_{B,t}$ is the stochastic innovation. This allows for a straightforward calculation of the moments. Assumption 2 ensures that the moments of the stochastic components $\mathbf{s}^{ex}_{B,t}$ are independent of the specific grid point $\mathbf{s}^{ex}_{A,t}$. Still, if Assumption 2 does not hold, the precomputation technique can be applied with minor modifications. Consider e.g. an extended model with stochastic volatility for the log productivity with

$$\begin{aligned} z_t &= \rho z_{t-1} + \exp(\sigma_t)\epsilon_t, \\ \sigma_t &= \bar{\sigma}(1-\rho) + \rho\sigma_{t-1} + \omega_t \end{aligned}$$

and standard normally distributed error terms ϵ_t and ω_t. The vector of exogenous state variables is the sum of $\mathbf{s}^{ex}_{A,t} = (\rho a_{t-1}, \bar{\sigma}(1-\rho) + \rho\sigma_{t-1})$ and $\mathbf{s}^{ex}_{B,t} = (\exp(\sigma_t)\epsilon_t, \omega_t)$. In this case computing

$$\mathrm{E}_t \prod_{i=1+d_{en}}^{3} \left(\mathcal{T}_{j_i}(\mathcal{A}_{i,t+1}) + \sum_{n=1}^{j_i} a_n f_n(\mathcal{A}_{i,t+1}, \mathcal{B}_{i,t+1}) \right)$$

yields

$$\begin{aligned} &\mathrm{E}_t \left[\left(\mathcal{T}_{j_2}(\mathcal{A}_{2,t+1}) + \sum_{n=1}^{j_2} a_n f_n(\mathcal{A}_{2,t+1}, \mathcal{B}_{2,t+1}) \right) \left(\mathcal{T}_{j_3}(\mathcal{A}_{3,t+1}) + \sum_{n=1}^{j_3} a_n f_n(\mathcal{A}_{3,t+1}, \mathcal{B}_{3,t+1}) \right) \right] \\ &= \mathcal{T}_{j_2}(\mathcal{A}_{2,t+1}) \mathcal{T}_{j_3}(\mathcal{A}_{3,t+1}) + \mathrm{E}_t \mathcal{C}(\mathcal{A}_{2,t+1}, \mathcal{A}_{3,t+1}, \mathcal{B}_{2,t+1}, \mathcal{B}_{3,t+1}), \end{aligned}$$

where the second term contains (cross) product terms and can be calculated numerically ex ante for a sub-grid of dependent variables.

5. Conclusion

I propose a precompuation technique, allowing to compute conditional expectations in DSGE models ex ante. The method is closely connected to Judd et al. (2011), but enhances their approach in several dimension. First, I state closed form and easy implementable solutions for the expectations over Chebyshev polynomials for any dimensions d. Second, I combine the precomputation technique with flexible polynomial functions as the Smolyak (1963) polynomial, tensor products polynomials or complete polynomials. Third and last, I show how the precomputation technique can be adopted to a framework with recursive preferences. The technique is adaptable for the vast majority of asset pricing models without imposing restrictive assumptions and yields an advantage that is likely to increase in the complexity of the models of interest.

Appendix

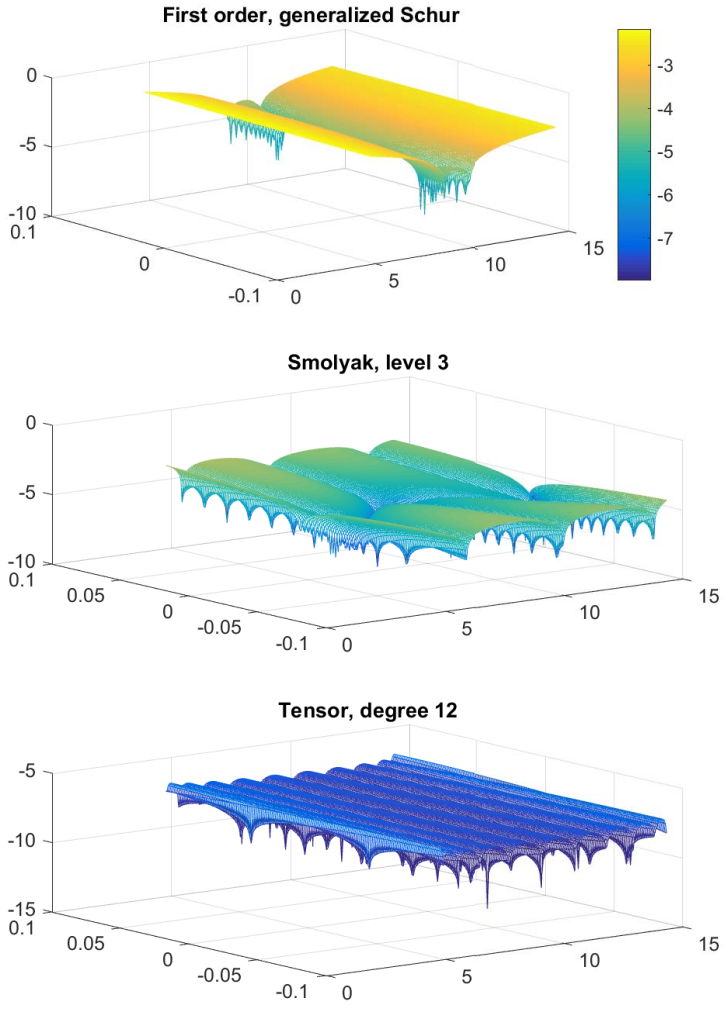

Figure 1: EEE for different solution types, complementary to Table 1

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