Pareto-optimality in Linear Public Goods Games

by

Sascha Hokamp[a] *and Michael Pickhardt*[b]

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Abstract: We derive a generalized method for calculating the *total* number of Pareto-optimal allocations (*NOPA*) in typical linear public goods games. Among other things, the method allows researchers to develop new experimental designs for testing the relevance of Pareto-optimality in experimental settings, for investigating alternative causes of the decline of voluntary contributions, or for analyzing the contribution behavior of the rich and poor in heterogeneous income settings. Further findings include that the *NOPA* is related to the marginal per capita return (*MPCR*) of a contribution to the public good and that the maximum number of free-riders tolerated by the Pareto-optimality concept is independent from the group size and income distribution. Finally, we apply our findings to a number of published linear public goods games, suggest an agenda for future research and provide a MATLAB code.

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[a]University of Muenster, The Muenster School of Business and Economics, 48143 Muenster/Germany, Email: saschahokamp@yahoo.de

[b]Corresponding author: Brandenburg University of Technology Cottbus, Faculty 3, Institute of Economics, Postbox 10 13 44, 03013 Cottbus/Germany, Email: michael@pickhardt.com

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1 Introduction

Public goods games and, in particular, linear public goods games are a popular tool for investigating human subject behavior. In fact, research during the last three decades has shed considerable light on why human subjects may voluntarily contribute to public goods. For example, Zelmer (2003) conducts a meta-analysis of linear public goods experiments and identifies eight variables that have statistically significant influence on average voluntary contributions to public goods. These variables include increasing marginal per capita returns ($MPCR$s), child subjects, communication (cheap talk), positive framing and constant groups for sessions (partners), which all have a positive influence. In contrast, heterogeneous endowments, experienced subjects and solicitating subject's beliefs about other participant's behavior before the start of the session have a negative influence on average voluntary contributions. Moreover, Ledyard (1995) and Cox and Sadiraj (2007) list a number of stylized facts that characterize voluntary contributions in linear public goods games. However, to our best knowledge the relevance of Pareto-optimality for maintaining voluntary contributions in such settings has not been investigated. This may be due to some unawareness that linear public goods games have several Pareto-optimal allocations and a lack of methods to determine the entire set of Pareto-optimal allocations in these games.

Therefore, it is the primary purpose of this paper to derive a generalized method for calculating the entire set of Pareto-optimal allocations in linear public goods games, where subjects may contribute voluntarily, face a discrete choice environment and cannot communicate with each other. Among other things, such a method could be used for investigating whether or not human subjects respond to the Pareto-optimality concept, for analyzing the decay of voluntary contributions, for studying the contribution behavior of the rich and poor in a heterogeneous income setting, for classifying linear public goods games and for an analysis of published linear public goods games, which we demonstrate in the course of the paper.

The paper is organized as follows. In the next section we briefly describe linear public goods games and offer a numerical motivation of our underlying idea. In sections 3 and 4 we develop a generalized method for calculating the *total* number of Pareto-optimal allocations ($NOPA$) in linear public goods games with homogeneous and heterogeneous settings, respectively. In addition, we derive several new figures, ratios and insights that are useful for analyzing linear public goods games and provide a MATLAB code for the generalized method. In section 5 we apply the calculation procedure to a number of published linear public goods experiments, which allows for analyzing Pareto-optimality and other aspects regarding these games. The final section concludes.

2 Model Design and Motivation

In this section we first discuss the standard model of linear public goods games in some detail and introduce several definitions with a view to prepare for a rigorous analysis in sections 3 and 4. Further, to simplify the analysis, we motivate our general idea with a small numerical example of a linear public goods game. In the course of the paper we will came back to this example on various occasions.

2.1 Linear Public Goods Games

Let the finite set of n agents be $\Im_n := \{1, \ldots, n\} \subset \mathbb{N}$. Let the quantity of the private good, which an agent $i \in \Im_n$ provides and consumes, be $y_i \in \mathbb{N}_0$. The total quantity X of the public good is given by,

$$X := \sum_{i=1}^{n} x_i, \tag{1}$$

where $x_i \in \mathbb{N}_0$ is the i-th agent's contribution to the public good. The public good is characterized by nonrivalness in consumption and, therefore, all n agents can simultaneously consume the total quantity of the public good, $X_i = X, \forall i \in \Im_n$ (see Samuelson 1954; Pickhardt 2006).

Each agent deals with a discrete choice environment and faces a linear payoff function $U : \mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{Q}^+$,

$$U(y_i, X) = \alpha y_i + \beta X, \qquad \forall i \in \Im_n, \tag{2}$$

and maximizes this payoff function subject to the budget constraint,

$$B = \gamma y_i + \delta x_i, \qquad \forall i \in \Im_n, \tag{3}$$

where parameters α, β, γ, and $\delta \in \mathbb{Q}^+$, and the budget $B \in \mathbb{N}$, $\forall i \in \Im_{\mathrm{n}}$, are constant and exogenously given in the underlying linear public goods experiment. Further, it is worth emphasizing that our analysis is concerned with either an one-shot game or with one particular round or trial of a finitely repeated non-cooperative n-person public goods game, where agents take their decisions simultaneously and independently. Put differently, we are interested in the set of alternatives an agent might have in a particular round, but not in the set of strategies an agent may pursue in linear public goods games.

Thus, the finite set of alternatives $\mathfrak{A}$, with $m \in \mathbb{N}$ elements, from which each agent can choose, is identical for each agent and determined by the budget $B \in \mathbb{N}$ and the smallest possible unit in which the budget may be spend, which we denote by $\epsilon \in \mathbb{N}$. Then, a discrete choice environment implies a functional relation between an agent's budget and the number of alternatives an agent may pursue in a round or trial of a linear public goods game. In general this functional form is,

$$m = \frac{B}{\epsilon} + 1, \qquad \forall i \in \Im_n. \tag{4}$$

In typical linear public goods experiments ϵ may represent one token. Note, however, if B and ϵ are measured in different scales (say tokens and token-cents), both must be expressed in terms of the smaller scale (here: token-cents). In addition, to ensure that m is an integer B, γ, and δ are restricted to values that allow for spending the entire budget, be it either on the private good, or on the public good or on a combination of both.

We denote by ζ and $\eta \in \mathbb{N}$ the smallest possible unit in which the quantity of the private and public good, respectively, may be produced or bought. By definition these parameters depend on γ and $\delta \in \mathbb{Q}^+$, respectively, and on $\epsilon \in \mathbb{N}$. Thus, we now define the production constraint as,

$$\epsilon = \gamma\zeta = \delta\eta, \qquad \forall i \in \Im_n, \tag{5}$$

which may be used for substituting ϵ in (4), so that either $\epsilon = \gamma\zeta$ or $\epsilon = \delta\eta$ yields the number of alternatives $m \in \mathbb{N}$, from which each agent may choose.

Yet, in most linear public goods games the private and public good parameters of the production constraint (5), which may be interpreted as per unit prices of the private and public good, respectively, are set equal to unity, i.e. $\gamma = \delta = 1$. Under these circumstances a prisoner's dilemma situation arises whenever the following condition holds (e.g. see Croson 2007, 200),

$$\frac{1}{n} < MPCR < 1, \tag{6}$$

where $MPCR := \beta/\alpha$ is the marginal per capita return of a contribution to the public good (see Isaac and Walker 1988, 182; Ledyard 1995, 149). In more general terms the necessary and sufficient condition for a prisoner's dilemma situation is,

$$\frac{1}{n} < \frac{\beta\gamma}{\alpha\delta} < 1. \tag{7}$$

A proof of condition (7) is provided in section 3.5, Proof 1. As in other linear public goods games (e.g. see Fehr and Gächter 2000, 982), we introduce an aggregate function for group payoff, denoted as welfare W, which is the sum of individual payoffs. Therefore, given our specifications according to (2) we obtain,

$$W = \alpha \sum_{i=1}^{n} y_i + n\beta X. \tag{8}$$

Furthermore, we regard W as a correspondence between two finite sets, in particular, the domain $\mathfrak{D}$, containing all feasible allocations, and the image $\mathfrak{W}$, containing all feasible levels of welfare. Thus, the number of allocations (NOA) and the number of welfare levels ($NOWL$) requires calculating how many elements $\mathfrak{D}$ and $\mathfrak{W}$, respectively, have. Therefore, we define,

$$NOA := |\mathfrak{D}| \tag{9}$$

and

$$NOWL := |\mathfrak{W}|\,, \tag{10}$$

where $|\ldots|$ denotes the number of elements of the relevant set. An allocation is said to be Pareto-optimal if and only if there exists no option in the set of feasible allocations that makes at least one agent better off in terms of its own individual payoff and, at the same time, none of the other agents worse off. The set of Pareto-optimal allocations $\mathfrak{P}$ is obviously a subset of domain $\mathfrak{D}$, and the *total* number of Pareto-optimal allocations ($NOPA$) requires calculating how many elements this subset has. Thus, we define,

$$NOPA := |\mathfrak{P}|\,. \tag{11}$$

The syntax introduced so far allows for defining a new ratio for comparing linear public goods games, henceforth denoted as the Pareto-ratio,

$$\text{Pareto-ratio} := \frac{NOPA}{NOA}. \tag{12}$$

Finally, we wish to emphasize that there are at least two scenarios of how to calculate NOA, $NOWL$, $NOPA$, and, therefore, the Pareto-ratio. These two scenarios depend on how to distinguish agents with respect to their individual contribution to the public good.

In the first scenario, following Pickhardt (2003; 2005), agents are indistinguishable if their individual contribution x_i to the public good is identical. Thus, in the first scenario the linear public goods game is characterized by the normal form,[1]

$$\mathfrak{F} = \{(\alpha; \beta; \gamma; \delta; B; m; n)\}. \tag{13}$$

In the second scenario agents are uniquely distinguishable, which yields the normal form,

$$\mathfrak{F} = \{\underbrace{(\alpha; \beta; \gamma; \delta; B; m; 1), \ldots, (\alpha; \beta; \gamma; \delta; B; m; 1)}_{n\text{-times}}\}. \tag{14}$$

In sections 3 and 4 we develop generalized calculation procedures for NOA, $NOWL$, $NOPA$, and the Pareto-ratio for both scenarios. However, before we turn to section 3, we proceed with a small numerical example concerning the first and second scenario.

2.2 Numerical Motivation

To motivate our general idea, we now consider a simple numerical case where a group of five agents, $n = 5$, which yields $\mathfrak{I}_5 = \{1, 2, 3, 4, 5\}$, simultaneously consumes a public good $X = X_i$, $\forall i \in \mathfrak{I}_5$, according to (1). Each agent faces the same set of parameters $\alpha = 4, \beta = 1$, which according to (2) yields $U(y_i, X) = 4y_i + X$, $\forall i \in \mathfrak{I}_5$, and $B = 2$, $\gamma = \delta = 1$, which according to (3) yields $2 = y_i + x_i$, $\forall i \in \mathfrak{I}_5$. In an experimental setting the variables B, U, y_i, x_i, X_i, and X are often measured in tokens during the experiment and, thereafter, total individual earnings in tokens are translated into cash by some predetermined exchange rate.

Moreover, each agent faces a discrete choice environment because each agent can spend the budget in units of one token only, which implies $\epsilon = 1$, $\forall i \in \mathfrak{I}_5$, so that (4) yields $|\mathfrak{A}| = m = 3$, $\forall i \in \mathfrak{I}_5$. Hence, each agent can choose in any round of the finitely repeated linear public goods game from a set of three alternatives, which are: full contribution $(FC) := (y_i = 0, x_i = 2)$, partial contribution $(PC) := (y_i = 1, x_i = 1)$, and non-contribution $(NC) := (y_i = 2, x_i = 0)$, and, thus $\mathfrak{A} := \{FC, PC, NC\}$ is the set of alternatives each agent is faced with.[2] Also, according to (5), $\gamma = \delta = \epsilon = 1$ yields $\zeta = \eta = 1$, $\forall i \in \mathfrak{I}_5$, and, therefore, the smallest possible units, in which the quantity of the private or public good may be produced or bought, coincide.

Furthermore, applying (6) yields $0.2 < 0.25 < 1$ and, thus, a prisoner's dilemma situation prevails. The Nash equilibrium solution is to contribute nothing to the public good, NC $(y_i = 2, x_i = 0)$, $\forall i \in \mathfrak{I}_5$, which yields $X = 0$. However, the resulting allocation is not Pareto-optimal. In fact, the group as a whole is better off if all group members contribute their entire endowment to the public good, FC $(y_i = 0, x_i = 2)$, $\forall i \in \mathfrak{I}_5$, which yields $X = 10$. Now the resulting allocation is Pareto-optimal, but it is worth noting that under the given circumstances this is not the only Pareto-optimal allocation. To visualize the entire set of Pareto-optimal allocations $\mathfrak{P}$ among the overall

[1] In more general terms the normal form of a linear public goods game is defined in section 4.1.

[2] Note that in finitely repeated linear public goods games with several rounds or trials agents may either choose a pure strategy or a mixed strategy, that is, they may choose in each round the same alternative or choose, for instance, each alternative with a probability $\pi_{i,\rho} > 0$, fulfilling the constraint $\sum_{\rho=1}^{3} \pi_{i,\rho} = 1$, respectively. But as noted, this kind of choice from the set of alternatives is of no relevance for the purpose of this paper, because all we need to consider are feasible allocations of one particular round of a finitely repeated linear public goods game or an one-shot game. This is because independently from the strategy an agent may have chosen, in any round each agent needs to select one of the feasible alternatives.

set of feasible allocations $\mathfrak{D}$, we use an extended version of Pickhardt's table (2003, 188; 2005, 147)[3].

Table 1: Set of Feasible Allocations and Pareto-optimal Allocations

Allocation	$n_{FC} \times U_{FC}$	$n_{PC} \times U_{PC}$	$n_{NC} \times U_{NC}$	X	Welfare	CA
1	—	—	5 × 8	0	40	0
2	—	1 × 5	4 × 9	1	41	4
3	—	2 × 6	3 × 10	2	42	9
4	1 × 2	—	4 × 10	2	42	4
5	—	3 × 7	2 × 11	3	43	9
6	1 × 3	1 × 7	3 × 11	3	43	19
7	—	4 × 8	1 × 12	4	44	4
8	1 × 4	2 × 8	2 × 12	4	44	29
9	**2 × 4**	—	**3 × 12**	**4**	**44**	**9**
10	—	5 × 9	—	5	45	0
11	1 × 5	3 × 9	1 × 13	5	45	19
12	**2 × 5**	**1 × 9**	**2 × 13**	**5**	**45**	**29**
13	1 × 6	4 × 10	—	6	46	4
14	**2 × 6**	**2 × 10**	**1 × 14**	**6**	**46**	**29**
15	**3 × 6**	—	**2 × 14**	**6**	**46**	**9**
16	**2 × 7**	**3 × 11**	—	**7**	**47**	**9**
17	**3 × 7**	**1 × 11**	**1 × 15**	**7**	**47**	**19**
18	**3 × 8**	**2 × 12**	—	**8**	**48**	**9**
19	**4 × 8**	—	**1 × 16**	**8**	**48**	**4**
20	**4 × 9**	**1 × 13**	—	**9**	**49**	**4**
21	**5 × 10**	—	—	**10**	**50**	**0**

Note: *Allocation denotes the number of allocation, n_{FC} (n_{PC}, n_{NC}) denotes the number of agents who choose alternative FC (PC, NC), respectively, where the constraint $n = 5 = n_{FC} + n_{PC} + n_{NC}$ is fulfilled and U_{FC}, (U_{PC}, U_{NC}) denotes individual payoff, which any agent who has selected alternative FC (PC, NC) will receive, respectively, $n_{FC} \times U_{FC}$ ($n_{PC} \times U_{PC}$, $n_{NC} \times U_{NC}$) denotes total payoff of the group of these agents, respectively (results not displayed), X denotes the total quantity of the public good, Welfare denotes overall payoff of the whole group, CA denotes the number of clone allocations, and allocations in bold denote Pareto-optimal allocations.*

Applying (8) yields $W = 4\sum_{i=1}^{5} y_i + 5X$, where $X = X_i$, $\forall i \in \mathfrak{I}_5$ is the quantity of the public good according to (1). In more general terms welfare is given by $W = n_{FC}U_{FC} + n_{PC}U_{PC} + n_{NC}U_{NC}$, where n_{FC} (n_{PC}, n_{NC}) denotes the number of agents who choose alternative FC (PC, NC), respectively, and the constraint $n = 5 = n_{FC}+n_{PC}+n_{NC}$ holds. Further, U_{FC} (U_{PC}, U_{NC}) denotes the individual payoff, which any agent who has selected FC (PC, NC) will receive, respectively.

We now consider Table 1 and suppose that assumptions of the first scenario hold, so that the normal form is $\mathfrak{F} = \{(4; 1; 1; 1; 2; 3; 5)\}$ according to (13). Thus, the domain of the correspondence W, that is the set of feasible allocations $\mathfrak{D}$, has $|\mathfrak{D}| = 21$ elements according to (9), which are shown in Table 1, ordered with respect to the level of welfare. Inspection of Table 1 shows that the image of the correspondence W, that is the set of feasible levels of welfare $\mathfrak{W}$, has $|\mathfrak{W}| = 11$ elements according to (10), which are 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50. The set of Pareto-optimal allocations $\mathfrak{P}$ has $|\mathfrak{P}| = 10$ elements according to (11), which are displayed in bold in Table 1. According to (12) the Pareto-ratio is $10/21 \approx 0.476$.[4] In particular, allocations 9, 12, 14, 15, 16, 17, 18, 19, 20, and 21 are Pareto-optimal, because in each case there exists no option in the

[3] Figure 2 in the Heldref online version of Pickhardt (2005) may differ in form and content from Figures 1 and 3. In this case, please consult the print version of Pickhardt (2005) in which Figure 2 is correct.

[4] Note that the number of elements of the set $\mathfrak{D}$ is calculated by $NOA = \frac{(n+m-1)!}{n!(m-1)!} = \frac{(5+3-1)!}{5!(3-1)!} = \frac{7!}{5!2!} = 21$. Calculation of the $NOPA$ is somewhat more complicated so that we refrain from presenting it here, but derive these calculation procedures in section 3.

set of feasible allocations $\mathfrak{D}$ that makes at least one agent better off in terms of its own individual payoff and, at the same time, none of the other agents worse off. For example, consider allocation 11, which is not Pareto-optimal. In this case there exists at least one option, for instance allocation 20, which makes at least one agent better off, here the agent who is on alternative FC and receives five tokens, without making any other agent worse off, because they can still get exactly the same payoffs as they get in allocation 11. Therefore, allocation 11 cannot be Pareto-optimal. In contrast, if we consider allocation 15, no such option exists in the set of feasible allocations $\mathfrak{D}$ and, thus, allocation 15 is Pareto-optimal.

We continue to consider Table 1, but now assume that the second scenario prevails, where agents are uniquely distinguishable and the normal form is $\mathfrak{F} = \{(4; 1; 1; 1; 2; 3; 1), \dots, (4; 1; 1; 1; 2; 3; 1)\}$, according to (14), with $n = 5$. In this case the set of feasible allocations $\mathfrak{D}$ is much larger and has $|\mathfrak{D}| = 243$ elements according to (9). To visualize this larger set in Table 1, we have introduced the number of clone allocations (CA) in the last column on the right hand side of Table 1. A clone allocation indicates that these allocations are not Pareto-distinguishable from their associated master allocation, where the latter is simply an arbitrarily chosen allocation from the set of feasible permutations (for details see section 3.4). Hence, the number of master allocations coincides with the number of allocations in the first scenario and we obtain NOA in Table 1 by adding up the number of clone allocations, which yields 222, plus the 21 master allocations, so that we get $222 + 21 = 243$. However, $NOWL$ is identical in both scenarios and $NOPA$ is obtained by adding up the clone allocations of the Pareto-optimal master allocations, which yields 121, plus the 10 Pareto-optimal master allocations, so that we get $121 + 10 = 131$. Thus, according to (12) the Pareto-ratio in the second scenario is $131/243 \approx 0.539$.[5]

Finally, with respect to the calculation of the Pareto-ratio it must be emphasized that the first scenario implies that each allocation shown in Table 1 occurs with the same probability. In contrast, regarding the second scenario the calculation implies that each agent may choose an alternative from the set of three alternatives with the same probability, i.e. $\pi_{i,FC} = \pi_{i,PC} = \pi_{i,NC} = 1/3, \forall i \in \mathfrak{I}_5$. Of course, it would be interesting to run an experiment with a view to see with which frequency each allocation actually occurs. Given a sufficient number of runs, the Pareto-ratio could then be recalculated based on the actual experimental results.

2.3 Some Further Aspects

First of all, it is worth emphasizing again that results derived so far are based on (i) the notion that agents take their decisions voluntarily and (ii) that any form of communication among agents during a round or trial is impossible, i.e., conditions which typically prevail in a standard linear public goods experiment. In contrast, if we allow for a benevolent social planner, who happens to have full information and the power to implement any allocation, the set of Pareto-optimal allocations $\mathfrak{P}$ would be reduced to allocation 21 (or 243) only. This is because from a social planner's group perspective, in any of the allocations 1 to 20 (or 1 to 242), at least one agent can be made better off without making any other agent worse off, by sharing the increase in welfare associated with a move from any of these allocations to allocation 21 (or 243).

[5]In this case the number of elements of the set $\mathfrak{D}$ is calculated by $NOA = m^n = 3^5 = 243$. Again, the calculation procedure for $NOPA$ is rather complex and, therefore, we refrain from displaying it here, but derive these calculation procedures in section 3.

Hence, a benevolent social planner would always use its power to implement allocation 21 (or 243).

The same result may emerge if we allow for communication among agents during a trial so that they can negotiate enforceable side payments with a view to implement allocation 21 (or 243). However, with respect to side payments at least two aspects must be taken into account. First, in voluntary contribution public goods games side payments do not always lead to Pareto-optimality (Pareto-efficiency) even when there are no transaction costs, complete information, and binding contracts (see Jackson and Wilkie 2005, 544). Second, even in cases where side payment contracts may lead to Pareto-optimality they must still be enforceable. Thus, we are left with a seemingly paradoxical first result.

Result 1: *Other things being equal, in linear public goods games voluntary contribution environments lead to a larger set of Pareto-optimal allocations* $\mathfrak{P}$ *than environments that allow for the use of force in one way or another.*

Also, given that a subset of the set of Pareto-optimal allocations is Pareto-optimal subject to the conditions mentioned above (e.g. allocations 9, 12, 14, 15, 16, 17, 18, 19, 20 of Table 1 in the first scenario, plus their clone allocations in the second scenario), one might follow Zeckhauser and Weinstein (1974, 644) and use the term 'mechanism-constrained Pareto-optimality' for this subset and the term 'unconstrained Pareto-optimality' for the remaining allocation, which is allocation 21 (or 243) of Table 1. In the following, however, we aim at calculating the entire set of Pareto-optimal allocations and it suffices to note that the distinction with respect to the two subsets can always be made regardless of the considered scenario.

Finally, Table 1 is also useful for illustrating the research aspects mentioned in the introduction in more detail. For example, it follows from Table 1 that for the given parameter setting the Pareto-optimality concept tolerates up to three deviants from alternative FC of full voluntary contribution (see allocations 9, 12, 14 and 16, first scenario, plus their clone allocations in the second scenario) and, thus, deviation of the majority of the group of five agents. Would the contributing subjects be prepared to tolerate such an allocation and continue to fully contribute, if they know that the allocation is Pareto-optimal? Alternatively, do subjects generally require that no other subject free rides on them by deviating from alternative FC, so that just allocation 21 (or 243) is tolerable? Do human subjects care at all about Pareto-optimality? Answering these and related questions would not only contribute evidence regarding the relevance of the Pareto-optimality concept for actual decisions of human subjects, but would also contribute to the ongoing research efforts on identifying motivations for voluntary contributions to public goods and their frequently observed decay in repeated games (see, among many others, Andreoni1995; Brandts and Schram 2001; Fischbacher et al. 2001; Masclet et al. 2003; Carpenter 2007; Figuières et al. 2009; Fischbacher and Gächter 2010; see also the survey by Chaudhuri 2011).

In the next two sections, we prepare the groundwork for answering these and related questions in experimental settings, where human subjects are faced with linear public goods games and for which researchers can freely choose all relevant parameters: B, m, n, α, β, γ, δ, ϵ, ζ, and η, subject to the aforementioned conditions.

3 Tracing Pareto-optimality: Homogeneous Settings

In this section we develop a generalized method to determine the number of Pareto-optimal allocations ($NOPA$) in homogeneous linear public goods games. In particular, we assume that subjects face a discrete choice environment, cannot communicate with each other during a trial, may contribute voluntarily, and have an identical payoff function, budget constraint, production constraint, and, therefore, the same $MPCR$. As in the previous section, we continue to distinguish two scenarios. To begin with, we consider the first scenario and assume that agents are restricted to a binary decision space. Next, we extend the method by allowing for any set of multiple alternatives and provide a graphical illustration of the calculation procedure. We then generalize the method by considering the second scenario. Finally, we prove necessary and sufficient conditions for identifying prisoner dilemma situations and Pareto-optimal allocations in homogeneous settings.

3.1 First Scenario: Binary Decision Space

Suppose each agent has just two alternatives, FC and NC, and faces an identical payoff function (2), budget constraint (3), and production constraint (5) (e.g. see McCorkle and Watts 1996, 235). Formally, we denote this as α, β, γ, and $\delta \in \mathbb{Q}^+$, $\epsilon = B \in \mathbb{N}$, ζ, and $\eta \in \mathbb{N}$, $\mathfrak{A} = \{FC, NC\}$, $|\mathfrak{A}| = m = 2$, $\forall i \in \mathfrak{I}_n$. As noted, with respect to the first scenario we assume agents are indistinguishable, if their individual contribution to the public good is identical, and, therefore, according to (13) the normal form of these games is $\mathfrak{F} = \{(\alpha; \beta; \gamma; \delta; B; 2; n)\}$.

Further, by using (5) we rewrite the two alternatives the i-th representative agent is faced with in general terms as, FC ($y_i = 0, x_i = \eta$) and NC ($y_i = \zeta, x_i = 0$), with $\zeta = B/\delta$ and $\eta = B/\gamma$. Again, the number of agents who have selected FC (NC) is denoted as n_{FC} (n_{NC}), respectively. According to (1), the total quantity X of the public good all n agents can simultaneously consume is $X_i = X = \sum_{i=1}^{n} x_i = \eta n_{FC} + 0 \cdot n_{NC} = \eta n_{FC}$, $\forall i \in \mathfrak{I}_n$. Based on (2) the individual payoff U_{FC} of full contributors, which are all agents $i \in \{\sigma(1), \ldots, \sigma(n_{FC})\} \subset \mathfrak{I}_n$, is given by,

$$U_{FC} := U_i(0, X) = \beta\, \eta\, n_{FC}, \tag{15}$$

where $\sigma \in \mathfrak{S}_n$ is a permutation and an element of the symmetric group of degree n.[6] The individual payoff U_{NC} of non-contributors, which are all agents $i \in \{\sigma(n_{FC} + 1), \ldots, \sigma(n)\} \subset \mathfrak{I}_n$, is given by,

$$U_{NC} := U_i\,(\zeta, X) = \alpha\, \zeta + \beta\, \eta\, n_{FC}, \tag{16}$$

where the second term on the right hand side of (16) is identical to (15) and indicates nonrival consumption of the public good. The total quantity of the private good consumed by the group of n agents is $\sum_{i=1}^{n} y_i = 0 \cdot n_{FC} + \zeta n_{NC} = \zeta n_{NC}$. Using $n = n_{FC} + n_{NC}$ allows us to replace n_{NC} by $n - n_{FC}$. Thus, applying (8) and rearranging yields an one-to-one correspondence that allows for calculating the level of welfare for each allocation,

$$W(n_{FC}) = \alpha\zeta n + (\beta\eta n - \alpha\zeta)\, n_{FC}. \tag{17}$$

[6]A permutation σ of integers between 1 and n is an one-to-one correspondence that assigns to each such integer another one. For example, the most simple one is the identity permutation $\iota(i) := i \;\; \forall i \in \mathfrak{I}_n$. According to Read (1972, 652-659) the product of two permutations is the permutation obtained by applying the two given permutations one after the other. The set of permutations, together with the operation above, forms the symmetric group of degree n, denoted as $\mathfrak{S}_n$.

Therefore, the maximum level of welfare is $W_{max} = W(n) = \beta\eta n^2$, where each agent has chosen FC. In addition, the minimum level of welfare amounts to $W_{min} = W(0) = \alpha\zeta n$, provided each agent has chosen NC. Furthermore, we regard W as an one-to-one correspondence between two finite sets, in particular, the domain $\mathfrak{D} := \{0, 1, \ldots, n\}$, containing all feasible values of n_{FC}, and the image $\mathfrak{W} := \{\alpha\zeta n, \alpha\zeta(n-1) + \beta\eta n, \ldots, \alpha\zeta + \beta\eta n(n-1), \beta\eta n^2\}$, containing all feasible levels of welfare. Then, applying (9) yields NOA and requires calculating how many elements $\mathfrak{D}$ has. Likewise, applying (10) yields $NOWL$ and requires calculating how many elements $\mathfrak{W}$ has. Thus, because of the one-to-one correspondence W between the two sets $\mathfrak{D}$ and $\mathfrak{W}$, calculating NOA coincides with calculating $NOWL$. Apparently, the set $\mathfrak{D}$ has $n + 1$ different elements and, therefore, we get,

$$NOA = |\mathfrak{D}| = |\mathfrak{W}| = NOWL = n + 1. \tag{18}$$

To give an example, consider Table 1 again. Because of $\mathfrak{A} = \{FC, NC\}$ and $|\mathfrak{A}| = m = 2, \forall i \in \mathfrak{I}_5$, Table 1 is *ceteris paribus* reduced to just the following six allocations: 1, 4, 9, 15, 19 and 21, representing the new set of feasible allocations $\mathfrak{D}$ which is now shown in Table 2, and applying (18) confirms the result. This notwithstanding, allocations 9, 15, 19 and 21 remain Pareto-optimal, representing the elements of the set of Pareto-optimal allocations $\mathfrak{P}$ and applying (11) yields the number of Pareto-optimal allocations $|\mathfrak{P}| = 4$. Hence, according to (12) the Pareto-ratio of the first scenario increases to $4/6 \approx 0.667$. Applying (13) yields $\mathfrak{F} = \{(4; 1; 1; 1; 2; 2; 5)\}$, which implies $\epsilon = \zeta = \eta = 2, \forall i \in \mathfrak{I}_5$. Then, applying (17) yields $W(n_{FC}) = 2n_{FC} + 40$. Thus, the maximum and minimum level of welfare is $W_{max} = W(5) = 50$ and $W_{min} = W(0) = 40$, respectively. These results are shown in Table 2 and for convenience the numbering of allocations in Table 2 corresponds to Table 1.

Table 2: Set of Feasible Allocations in the Binary Decision Case

Allocation	$n_{FC} \times U_{FC}$	$n_{NC} \times U_{NC}$	X	Welfare	CA
1	—	5×8	0	40	0
4	1×2	4×10	2	42	4
9	$\mathbf{2 \times 4}$	$\mathbf{3 \times 12}$	**4**	**44**	**9**
15	$\mathbf{3 \times 6}$	$\mathbf{2 \times 14}$	**6**	**46**	**9**
19	$\mathbf{4 \times 8}$	$\mathbf{1 \times 16}$	**8**	**48**	**4**
21	$\mathbf{5 \times 10}$	—	**10**	**50**	**0**

Note: *Allocation denotes the number of allocation corresponding to Table 1, n_{FC} (n_{NC}) denotes the number of agents who choose alternative FC (NC), respectively, where the constraint $n = 5 = n_{FC} + n_{NC}$ is fulfilled and U_{FC}, (U_{NC}) denotes the individual payoff, which any agent who has selected FC (NC) will receive, respectively, $n_{FC} \times U_{FC}$ ($n_{NC} \times U_{NC}$) denotes total payoff of the group of these agents, respectively (results not displayed), X denotes the total quantity of the public good, Welfare denotes overall payoff of the whole group, CA denotes the number of clone allocations, and allocations in bold denote Pareto-optimal allocations.*

Inspection of Table 2 also reveals that allocations 9, 15 and 19 must belong to the set of Pareto-optimal allocations $\mathfrak{P}$ simply because in each case at least one agent receives an individual payoff (i.e., 12, 14 or 16 tokens, respectively), which is higher than the individual payoff he or she would get if allocation 21 prevails, i.e., ten tokens (see Pickhardt 2003, 188; Pickhardt 2005, 147-148). Therefore, the general idea of how to calculate the $NOPA$ is to use the individual payoff associated with the allocation that yields the maximum level of welfare as a benchmark. For example, in Table 1 or Table 2, the individual payoff of ten tokens in allocation 21 would be that benchmark. Hence, in the binary case calculating the $NOPA$ just requires counting the number of allocations in which at least one agent receives a higher individual payoff than the benchmark, plus adding one for the benchmark allocation, which is always Pareto-optimal.

To proceed, we now generalize the calculation procedure for $NOPA$, with respect to the binary case of the first scenario. If every agent contributes to the public good, $n_{FC} = n$ applies, the allocation with the maximum level of welfare $W_{max} = W(n)$ occurs and the individual payoff of each agent is the benchmark, which we denote in general terms by using (15) as,

$$\widehat{U}_{FC} \coloneqq \beta\, \eta\, n, \qquad \forall i \in \Im_n. \tag{19}$$

As noted, the necessary and sufficient condition for Pareto-optimal allocations is that a non-contributor gains a higher individual payoff (16) than the benchmark (19), which yields,

$$\alpha\, \zeta + \beta\, \eta\, n_{FC} > \beta\, \eta\, n. \tag{20}$$

Note that on purely formal grounds condition (20) also applies for $n_{FC} = n$ and, therefore, includes the benchmark allocation. For convenience a proof of (20) is provided in section 3.5, Proof 2.

Rearranging (20), substituting of ζ/η by δ/γ according to (5) and using the ceiling function[7] yields,

$$n_{FC} \geq n - \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil + 1. \tag{21}$$

We now define the right hand side of (21) as minimum number of full contributors, which is necessary and sufficient for Pareto-optimality,

$$n_{min} \coloneqq n - \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil + 1. \tag{22}$$

Moreover, the minimum number of full contributors n_{min} is related to the maximum number of free-riders n_{max}, which the Pareto-optimality concept just tolerates. Thus, we get,

$$n_{max} \coloneqq n - n_{min} = \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil - 1. \tag{23}$$

Note that n_{max} does not depend on the budget B or the number of agents n. Thus, other things being equal, an increase of n requires an increase of n_{min} and *vice versa* according to (23). Apparently, for the binary numerical case (see Table 2), the minimum number of full contributors is $n_{min} = 2$ according to (22), which leads to the maximum number of free-riders $n_{max} = 3$ just tolerated by the Pareto-optimality concept, according to (23). Recall, however, that in this paper the term 'free-rider' refers to those agents who do not fully contribute, although, in the binary case this coincides with non-contributing.

Next, we introduce the number of allocations with k free-riders, which for the binary case yields,

$$NOA(k) = 1, \qquad \forall k \in \{0, 1, \ldots, n\}. \tag{24}$$

Thus, calculating $NOPA$ requires adding the number of allocations with up to n_{max} free-riders, and, therefore, we get,

$$NOPA = \sum_{k=0}^{n_{max}} NOA(k) = \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil. \tag{25}$$

[7] Read (1972, 66) defines the integral part of $a \in \mathbb{R}$, denoted as $\lfloor a \rfloor$, that is, the largest integer not greater than a. For example, we have $\lfloor 5 \rfloor = 5$, $\lfloor -4.2 \rfloor = -5$, $\lfloor 4.2 \rfloor = 4$, $\lfloor 1.618 \rfloor = 1$, and so on. The integral is also called floor function. The ceiling function expressed by the integral part (or floor function) is $\lceil a \rceil := -\lfloor -a \rfloor$, that is, the smallest integer not less than a. For example, we get $\lceil 5 \rceil = 5$, $\lceil -4.2 \rceil = -4$, $\lceil 4.2 \rceil = 5$, $\lceil 1.618 \rceil = 2$, and so on.

Equation (25) allows for calculating the $NOPA$ in any linear public goods game, provided the assumptions we have made in this subsection apply. For example, according to (18), in the reduced numerical example (see Table 2), there are six allocations, each having an unique level of welfare and (25) reveals that four of these allocations are Pareto-optimal. Therefore, we can identify allocations 9, 15, 19, and 21 as Pareto-optimal, which formally yields $NOA(3) = 1$, $NOA(2) = 1$, $NOA(1) = 1$, and $NOA(0) = 1$, respectively, so that we get $NOPA = 4$. Inspection of Table 2 confirms these results. Hence, we are now in the position to summarize our findings.

Result 2: *Let the parameters n, α, β, γ, and δ be selected such that condition (7) holds, so that a prisoner's dilemma prevails, and let the normal form of the binary linear public goods game be $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B;2;n)\}$, according to (13), implying $B = \epsilon$. Then, the number of allocations (NOA) is identical to the number of welfare levels ($NOWL$) and is calculated from (18), which is*

$$NOA = NOWL = n + 1,$$

and the number of Pareto-optimal allocations ($NOPA$) is calculated from (25), which is

$$NOPA = \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil.$$

3.2 First Scenario: Multiple Decision Space

The result of the preceding subsection is already useful for some linear public goods games, but in the vast majority of these games agents may pursue three or more alternatives in one particular round or trial. Therefore, we now extend the binary decision space to a multiple decision space, where each agent faces a finite set of two or more alternatives. Yet, we continue to assume that the payoff function, the budget constraint, and the production constraint is identical for each agent. Again, according to the first scenario agents are indistinguishable, if their individual contribution to the public good is identical.

3.2.1 Preliminaries

We denote the multiple decision space formally as α, β, γ, and $\delta \in \mathbb{Q}^+$, $\epsilon \le B \in \mathbb{N}$, ζ, η and $m \in \mathbb{N}$, which yields $|\mathfrak{A}| = m \ge 2$, $\forall i \in \mathfrak{I}_n$ and, therefore, according to (13) the normal form is $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B;m;n)\}$. In addition, by substituting (4) into (5) and rearranging, we get,

$$\zeta = \frac{B}{\gamma(m-1)}, \qquad \forall i \in \mathfrak{I}_n, \tag{26}$$

and,

$$\eta = \frac{B}{\delta(m-1)}, \qquad \forall i \in \mathfrak{I}_n. \tag{27}$$

To simplify notation, we rewrite the m alternatives the representative i-th agent is faced with, in general terms as, $PC_j := (y_i = \zeta j,\ x_i = \eta(m - j - 1))$, where $j \in \{0, 1, \ldots, m-1\} := \mathfrak{I}_m$ denotes the $j+1$-th alternative and the entire set of partial contribution alternatives is ordered from full contribution, $j = 0$, to non-contribution, $j = m-1$. We denote this set of alternatives formally as $\mathfrak{A} = \{PC_0, PC_1, \ldots, PC_{m-1}\}$, $\forall i \in \mathfrak{I}_n$.

Hence, the binary decision case of the preceding subsection is included in this framework by allowing agents to select only between two alternatives, where FC is equivalent to PC_0, and NC is equivalent to PC_1, with $m = 2$, $\forall i \in \mathfrak{I}_n$ (see Table 2). Further, the numerical case of section 2.2 is included by regarding FC (PC, NC) as PC_0 (PC_1, PC_2), respectively, with $|\mathfrak{A}| = m = 3$, $\forall i \in \mathfrak{I}_5$ (see Table 1). In particular, the new notation allows for interpreting the subscript $j \in \mathfrak{I}_m$ as a multiple of a fraction of the budget, in terms of ζ, that is not contributed to the public good. For instance, in the numerical case of section 2.2, where $\gamma = 1$, $B = 2$ and $m = 3$, we get $\zeta = 1$, according to (26). It follows by definition of PC_j given above that PC_0 (PC_1, PC_2) refers to the i-th agent's private good consumption, $y_i = 1 \cdot 0 = 0$ ($y_i = 1 \cdot 1 = 1$, $y_i = 1 \cdot 2 = 2$), respectively. Therefore, in general PC_j, $j \in \mathfrak{I}_m$, refers to $y_i = \zeta j$, as noted above. This makes it clear that the index j, with $j > 0$, indicates free riding behavior, that is, the agent chooses not to contribute the entire budget to the public good.

3.2.2 NOA and NOWL Calculation Procedures

To proceed, we now calculate NOA and $NOWL$ in the multiple decision space. We begin with by denoting the number of agents who have selected alternative PC_j as $n_j \in \{0, 1, \ldots, n\}$, where $j \in \mathfrak{I}_m$. Thus, based on (1) the total quantity X of the public good, which all n agents can simultaneously consume, is $X_i = X = \sum_{i=1}^{n} x_i = \sum_{j=0}^{m-1} \eta n_j(m - j - 1)$, $\forall i \in \mathfrak{I}_n$. Note that the last term on the right hand side equals $\sum_{j=0}^{m-2} \eta n_j\ (m - j - 1)$, because the contribution of agents who have selected PC_{m-1} (non-contributors or full free-riders) is zero. By applying (2), the individual payoff of full contributors, U_0, amounts to,

$$U_0 \coloneqq U_i(0, X) = \beta \sum_{j=0}^{m-2} \eta n_j(m - j - 1), \tag{28}$$

where full contributors are agents who have selected alternative PC_0, that is, all agents $i \in \{\tau(1), \ldots, \tau(n_0)\} \subset \mathfrak{I}_n$, with $\tau \in \mathfrak{S}_n$ denoting a permutation. The individual payoff of agents who contribute part of their budget to the public good, U_k, is given by,

$$U_k \coloneqq U_i(\zeta k, X) = \alpha\zeta k + \beta \sum_{j=0}^{m-2} \eta n_j(m - j - 1), \tag{29}$$

where partial contributors are agents who have selected alternative PC_k, $k \in \{1, \ldots, m-2\} \subset \mathfrak{I}_m$, that is, all agents $i \in \{\tau(\sum_{j=0}^{k-1} n_j + 1), \ldots, \tau(\sum_{j=0}^{k} n_j)\} \subset \mathfrak{I}_n$. Finally, the individual payoff of non-contributors, U_{m-1}, is given by,

$$U_{m-1} \coloneqq U_i(\zeta(m-1), X) = \alpha\zeta(m-1) + \beta \sum_{j=0}^{m-2} \eta n_j(m - j - 1), \tag{30}$$

where non-contributors are agents who have selected alternative PC_{m-1}, that is, all agents $i \in \{\tau(\sum_{j=0}^{m-2} n_j + 1), \ldots, \tau(n)\} \subset \mathfrak{I}_n$.

Moreover, using $n = \sum_{j=0}^{m-1} n_j$ allows us to remove one of the variables $n_0, n_1, \ldots, n_{m-2}, n_{m-1}$ and we replace n_{m-1} by $n - \sum_{j=0}^{m-2} n_j$. Thus, according to (28), (29) and (30) the total quantity of the private good consumed by all n agents is $\sum_{i=1}^{n} y_i = \sum_{j=0}^{m-1} \zeta\ j\ n_j = \sum_{j=0}^{m-2} \zeta\ j\ n_j + \zeta\ (m-1)(n - \sum_{j=0}^{m-2} n_j) = \zeta(m-1)n - \sum_{j=0}^{m-2} \zeta n_j(m - j - 1)$. Applying (8) yields an onto correspondence[8] that allows for calculating the level of welfare for each allocation,

$$W(n_0, n_1, \ldots, n_{m-3}, n_{m-2}) = \alpha\zeta(m-1)n + (\beta\eta n - \alpha\zeta) \sum_{j=0}^{m-2} n_j \,(m-j-1). \tag{31}$$

Thus, the maximum level of welfare is $W_{max} = W(n, 0, \ldots, 0) = \beta\eta(m-1)n^2$, where each agent has selected alternative PC_0, full contribution. Furthermore, the minimum level of welfare amounts to $W_{min} = W(0, \ldots, 0) = \alpha\zeta(m-1)n$, provided that each agent has selected non-contribution PC_{m-1}. As proposed in section 2.1, we now regard W as an onto correspondence between two finite sets, in particular, the domain $\mathfrak{D} := \{(n_0, n_1, \ldots, n_{m-3}, n_{m-2}) \in \mathbb{N}_0^{m-1} \mid \sum_{j=0}^{m-2} n_j \leq n\}$, containing all feasible values of $n_0, n_1, \ldots, n_{m-3}, n_{m-2}$, and the image $\mathfrak{W} := \{\alpha\zeta(m-1)n,\ \alpha\zeta((m-1)n-1) + \beta\eta n, \ldots,$ $\alpha\zeta + \beta\eta((m-1)n-1)n, \beta\eta(m-1)n^2\}$, containing all feasible levels of welfare. Again, applying (9) requires calculating how many elements $\mathfrak{D}$ has, and applying (10) requires calculating how many elements $\mathfrak{W}$ has. Using combinatorics[9] we obtain,

$$NOA = |\mathfrak{D}| = \binom{n+m-1}{n} = \frac{(n+m-1)!}{n!(m-1)!} \tag{32}$$

and,

$$NOWL = |\mathfrak{W}| = (m-1)n + 1. \tag{33}$$

To give an example, we consider Table 1 again. Based on the normal form $\mathfrak{F} = \{(4; 1; 1; 1; 2; 3; 5)\}$, (26) and (27), we get $\epsilon = \zeta = \eta = 1$ and, therefore, $W(n_0, n_1) = 2n_0 + n_1 + 40$ according to (31). Hence, the maximum and minimum level of welfare is $W_{max} = W(5, 0) = 50$ and $W_{min} = W(0, 0) = 40$, respectively. Applying (32) and (33) with $m = 3$ and $n = 5$ yields $NOA = 21$ and $NOWL = 11$. Inspection of Table 1 confirms these results.

3.2.3 NOPA Calculation Procedure

To proceed, we slightly modify the technique already employed in the binary decision space. As noted, if every agent fully contributes to the public good, which implies $n_0 = n$, each agent gets the benchmark because the allocation with the maximum level of welfare occurs, that is, $W_{max} = W(n, 0, \ldots, 0)$. Substituting this information into (28) yields,

$$\widehat{U_0} := \beta\,\eta\,(m-1)\,n, \qquad \forall i \in \mathfrak{I}_n, \tag{34}$$

which is a generalization of (19). Next we aim at identifying allocations that might belong to the set of Pareto-optimal allocations in the multiple decision space. Therefore, the necessary, though not sufficient, condition for a Pareto-optimal allocation is that

[8]According to Read (1972, 570) a correspondence between two sets is called an onto correspondence (or surjective correspondence), if to every element of the domain corresponds exactly one element of the image and every element of the image corresponds to at least one element of the domain.

[9]The number of elements of the set $\mathfrak{D}$ coincides with the number of elements of the set $\widehat{\mathfrak{D}} := \{(n_0, n_1, \ldots, n_{m-3}, n_{m-2}, n_{m-1}) \in \mathbb{N}_0^m \mid \sum_{j=0}^{m-1} n_j = n\}$, because there exists an one-to-one correspondence between these two sets. Therefore, with respect to Read (1972, 109-112) we may describe the problem as that of choosing n objects (agents) from m sets of objects (agents who have selected one of m possible alternatives). Hence, we apply the general formula for selections with repetitions.

non-contributing agents must receive a higher individual payoff (30) than the benchmark (34), which yields,

$$\alpha\,\zeta\,(m-1)+\beta\,X>\beta\,\eta\,(m-1)\,n. \tag{35}$$

Rearranging (35) yields the necessary condition,

$$X>(\eta\,n-\frac{\alpha}{\beta}\,\zeta)(m-1). \tag{36}$$

Next, we define the right hand side of (36) as the minimum quantity of the public good, which must be exceeded in order to allow for Pareto-optimality,

$$X_{min}:=(\eta\,n-\frac{\alpha}{\beta}\,\zeta)(m-1). \tag{37}$$

Hence, allocations where the quantity of the public good exceeds the minimum quantity X_{min} might belong to the set of Pareto-optimal allocations. Put differently, allocations where the quantity of the public good does not exceed the minimum quantity X_{min} cannot be Pareto-optimal. Thus, for convenience the necessary condition (36) may be written as,

$$X>X_{min}. \tag{38}$$

Applying (37) to the parameter set of the numerical case of section 2.2 yields $X_{min}=2$ and, therefore, according to (38) the necessary condition is $X>2$. Hence, the set of quantities of the public good is $\{3,4,5,6,7,8,9,10\}$, and each of these quantities might be associated with Pareto-optimal allocations. Inspection of Table 1 confirms that this set indeed contains the entire set of quantities of the public good, $\{4,5,6,7,8,9,10\}$, which are associated with Pareto-optimal allocations. In general terms (38) yields,

$$\sum_{j=0}^{m-2}\eta n_j(m-j-1)>(\eta\,n-\frac{\alpha}{\beta}\,\zeta)(m-1). \tag{39}$$

Division by η, substitution of ζ/η by δ/γ according to (5), using the ceiling function and rearranging yields,

$$\sum_{j=0}^{m-2}n_j(m-j-1)\geq(m-1)n-\left\lceil\frac{\alpha\delta(m-1)}{\beta\gamma}\right\rceil+1. \tag{40}$$

Moreover, we define,

$$NOWL_m:=\left\lceil\frac{\alpha\delta(m-1)}{\beta\gamma}\right\rceil, \tag{41}$$

which is the number of welfare levels potentially associated with Pareto-optimal allocations, where each agent may choose from an identical set of m alternatives. Note that the remaining terms on the right hand side of (40) coincide with the $NOWL$ according to (33). Hence, the parameter setting of the numerical case presented in section 2.2 yields $NOWL_3=8$, according to (41). Then, by using the inverse of the correspondence W, given by (31), we obtain an approximation for the upper bound of $NOPA$,

$$NOPA\leq|W^{-1}(\{\beta\eta(m-1)n^2-(\beta\eta n-\alpha\zeta)(NOWL_m-1),\ldots,\beta\eta(m-1)n^2\})|. \tag{42}$$

The next step in calculating $NOPA$ requires searching for the inverse image of $NOWL_m$-levels of welfare, that is, solving $NOWL_m$-equations with non-negative integers. In general, these equations are given by,

$$\begin{array}{rcl}(m-1)n_0+(m-2)n_1+\cdots+2n_{m-3}+n_{m-2} & = & (m-1)n \\ (m-1)n_0+(m-2)n_1+\cdots+2n_{m-3}+n_{m-2} & = & (m-1)n-1 \\ \vdots & & \vdots \\ (m-1)n_0+(m-2)n_1+\cdots+2n_{m-3}+n_{m-2} & = & (m-1)n-NOWL_m+2 \\ (m-1)n_0+(m-2)n_1+\cdots+2n_{m-3}+n_{m-2} & = & (m-1)n-NOWL_m+1.\end{array} \tag{43}$$

For example, with respect to the numerical case of section 2.2, we get $NOPA \leq |W^{-1}(\{43, 44, 45, 46, 47, 48, 49, 50\})| = 17$, where $|\{43, 44, 45, 46, 47, 48, 49, 50\}| = NOWL_3$.

However, in addition to the necessary condition (38), we need a sufficient condition that allows for verifying whether or not a solution belongs to the set of Pareto-optimal allocations $\mathfrak{P}$. This sufficient condition is the minimum number of full contributors necessary for Pareto-optimal allocations. We obtain the condition by solving (39) for $n_1 = n_2 = \ldots = n_{m-2} = 0$, that is, using $\sum_{j=0}^{m-2} \eta n_j(m-j-1) = \eta n_0(m-1) > (\eta\, n - \frac{\alpha}{\beta}\, \zeta)(m-1)$ to get the constraint,

$$n_0 \geq n - \frac{\alpha\delta}{\beta\gamma} + 1. \tag{44}$$

By making use of the ceiling function we obtain n_{min} according to (22). Note that this is the same sufficient condition as in the binary decision space. A formal proof of the necessary and sufficient conditions is provided in section 3.5, Proof 2 and Proof 3. Again, the minimum number of full contributors is related to the maximum number of free-riders, given by (23), which the Pareto-optimality concept just tolerates. Recall, however, that in this paper free-riders are defined as those who do not choose the alternative of contributing their entire budget to the public good (full contribution, PC_0 or FC).

To continue, we now extend (24) of the preceding section by using combinatorics[10], which yields the number of allocations with k free-riders,

$$NOA(k) = \frac{(k+m-2)!}{k!(m-2)!}, \qquad \forall k \in \{0, 1, \ldots, n\}. \tag{45}$$

Then, according to (11) we eventually obtain,

$$NOPA = \sum_{k=0}^{n_{max}} NOA(k) = \begin{pmatrix} n_{max}+m-1 \\ n_{max} \end{pmatrix} = \frac{(n_{max}+m-1)!}{n_{max}!(m-1)!}. \tag{46}$$

To give an example, consider the numerical case of section 2.2 again, with $\mathfrak{F} = \{(4; 1; 1; 1; 2; 3; 5)\}$. By applying (43), we get the following eight equations,

$$\begin{array}{rcl} 2n_0+n_1 & = & 10 \\ 2n_0+n_1 & = & 9 \\ \vdots & & \vdots \\ 2n_0+n_1 & = & 4 \\ 2n_0+n_1 & = & 3. \end{array} \tag{47}$$

According to (22) the minimum number of full contributors is $n_{min} = 2$ and according to (23) the maximum number of free-riders is $n_{max} = 3$, which both coincide with the results of the binary case. Applying (46) yields $NOPA = 10$ and inspection of Table 1 confirms this result, because allocations 9, 12, 14, 15, 16, 17, 18, 19, 20, and 21 represent the set of Pareto-optimal allocations $\mathfrak{P}$. Finally, applying (12) yields the Pareto-ratio of this public goods game, with $10/21 \approx 0.476$.

[10]We apply the general formula for selections with repetition as in (32).

3.2.4 Results for NOA, NOWL, and NOPA Calculation Procedures

By making some modifications to the binary case, we were able to obtain calculation procedures for NOA, $NOWL$, and $NOPA$ in the multiple decision space of the first scenario. The results may be summarized as follows.

Result 3: *Let the parameters n, α, β, γ, and δ be selected such that condition (7) holds, so that a prisoner's dilemma prevails, and let the normal form of the linear public goods game be $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B;m;n)\}$, according to (13). Then, the number of allocations (NOA) is calculated from (32), which is*

$$NOA = \frac{(n+m-1)!}{n!(m-1)!},$$

the number of welfare levels (NOWL) is calculated from (33), which is

$$NOWL = (m-1)n + 1,$$

and the number of Pareto-optimal allocations (NOPA) is calculated from (46), which is

$$NOPA = \frac{(n_{max}+m-1)!}{n_{max}!(m-1)!},$$

where the maximum number of free-riders (n_{max}) just tolerated by the Pareto-optimality concept is calculated from (23), which is $\left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil - 1$.

Moreover, regarding the frequently encountered case of $\epsilon = 1$, it follows from (4) that $m = B + 1$ and, thus, $m - 1 = B$, which allows for simplifying (32), (33) and (46) accordingly. Likewise, for all $MPCR$s fulfilling $0.5 \leq MPCR < 1$, according to (6) and (7), it follows from (23) that $n_{max} = 1$ and, thus, (46) can be simplified to $NOPA = m$, where m is calculated from (4).

Result three can be readily applied to a large number of linear public goods games. Besides, it is worth emphasizing that condition (44) allows for an identification of Pareto-optimal allocations in real time, simply by counting the number of full contributors. Hence, during an experimental session, in any round or trial of a computerized linear public goods game it is possible to show on the subject's screen an information that indicates whether or not the present allocation is Pareto-optimal. This greatly simplifies any experimental testing of the Pareto-optimality concept.

To complement our analysis, in the next section we offer a graphical interpretation of the calculation procedures introduced so far.

3.3 First Scenario: Graphical Illustration

In this section we provide a graphical illustration of the calculation procedures for NOA, $NOWL$ and $NOPA$. In particular, we base the illustration on the numerical example of section 2.2, subject to the assumptions of the first scenario and the restriction $m = 3$. Thus, according to (13) the normal form is $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B;3;n)\}$, which implies $2\epsilon = B \in \mathbb{N}$ according to (4). As noted, in this case $n_0 + n_1 + n_2 = n$ prevails, where n_0 (or n_{FC}) is the number of full contributors, n_1 (or n_{PC}) is the number of partial contributors and n_2 (or n_{NC}) is the number of non-contributors. Moreover, we can remove one of the variables n_0, n_1, n_2 and we replace n_2 by $n - n_0 - n_1$. Therefore, the graphical representation has two dimensions, that are the number of full contributors

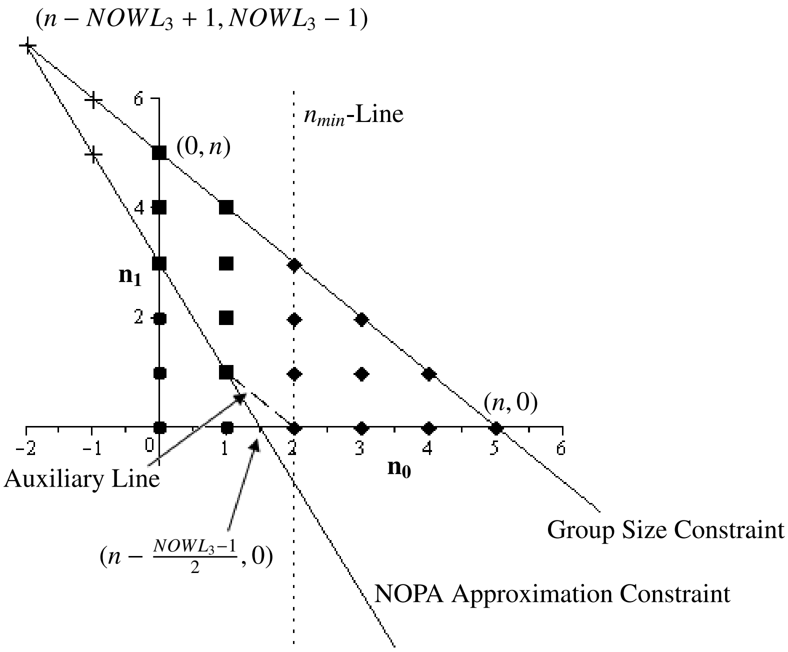
n_0 and the number of partial contributors n_1. This makes it clear that in Figure 1 the graphical interpretation is restricted to the number of agents shown in Table 1 and does not include any payoffs as in Table 1. This notwithstanding, lattice points represent feasible allocations in the underlying linear public goods game, which are denoted by (n_0, n_1). For example, the Nash equilibrium, allocation 1 in Table 1, is represented by the origin $(0, 0)$ and marked by a dot in Figure 1. Likewise, allocation 21 of Table 1 is located at $(5, 0)$ and marked by a diamond.

The general idea of graphically solving the calculation problem is given by Pick's formula for lattice points of lattice polygons (see Pick 1899; Funkenbusch 1974; Grünbaum and Shepard 1993).[11] However, for cases considered in this paper it is sufficient to use the notation of Hadwiger and Wills (1975, 63, Eq. 1.4), that is,

$$G = F + \frac{\hat{G}}{2} + 1, \tag{48}$$

where G is the number of lattice points, F is the area and $\hat{G}$ is the number of boundary lattice points of the lattice polygon.

Figure 1: Graphical Representation of Table 1 (NOA, NOWL, NOPA)



We begin with calculating NOA, which is the sum of lattice points of the triangle Δ_1 with vertices $(0, 0)$, $(n, 0)$ and $(0, n)$, that is, the triangle given by the two axis and the

[11]Georg Pick was born in 1859 in Vienna and was Professor of Mathematics at the German section of the Charles University, Prague. He was a member of the committee responsible for offering Albert Einstein a position in Prague and later became a friend of Einstein. Pick also contributed the mathematical appendix to Albert Weber's work 'Theory of the Location of Industries'. He retired in 1929 and went back to Vienna. In 1938 he returned to Prague and in 1942 he was deported from Prague to the concentration camp 'Theresienstadt', where he died two weeks after his arrival on July 26, 1942, at the age of 82 (see Fritsch (2001) and http://en.wikipedia.org/wiki/Georg_Alexander_Pick accessed February 28, 2011).

group size constraint. The latter is simply the line that links $(n, 0)$ and $(0, n)$. Applying (48) yields $F = n^2/2$ and $\hat{G} = 3n$, because each of the three boundary edges has essentially n lattice points.[12] Thus, in compliance with (32) we obtain,

$$NOA = \frac{n^2}{2} + \frac{3n}{2} + 1 = \frac{(n+2)(n+1)}{2} = \frac{(n+2)!}{n!2!} = \begin{pmatrix} n+2 \\ n \end{pmatrix}. \tag{49}$$

Hence, in Figure 1 NOA is illustrated by lattice points marked by dots, boxes and diamonds, which sum up to the 21 feasible allocations of Table 1.

To calculate $NOWL$ we have to find one representative allocation for each feasible level of welfare. For example, lattice points located on the edges $(0, 0)$, $(0, n)$ and $(0, n)$, $(n, 0)$ always have this property. This is because these relevant lattice points allow for reproducing the second term of (31), where the first factor, $\beta\eta n - \alpha\zeta$, represents the difference in welfare between two neighboring lattice points, and, for $B = 2\epsilon$, the second factor, $\sum_{j=0}^{m-2} n_j(m - j - 1)$, becomes $2n_0 + n_1$, which corresponds to (47) or (52). Thus, in compliance with (33) just counting lattice points on these edges gives,

$$NOWL = 2n + 1, \tag{50}$$

which yields 11 feasible different levels of welfare as in Table 1.

With respect to the $NOPA$, we first consider the approximation for the upper bound of the $NOPA$, which is calculated from (42), and leads to,

$$NOPA \leq \left| W^{-1}(\{2\beta\eta n^2 - (\beta\eta n - \alpha\zeta)(NOWL_3 - 1), \ldots, 2\beta\eta n^2\}) \right|, \tag{51}$$

where according to (41) $NOWL_3 = \left\lceil \frac{2\alpha\delta}{\beta\gamma} \right\rceil \in \mathbb{N}$, which is equal to eight in the numerical example of Table 1. According to (43), we get $NOWL_3$-equations with non-negative integers,

$$\begin{array}{rcc} 2n_0 + n_1 & = & 2n \\ 2n_0 + n_1 & = & 2n - 1 \\ \vdots & & \vdots \\ 2n_0 + n_1 & = & 2n - NOWL_3 + 1. \end{array} \tag{52}$$

The first equation from above, that is $2n_0 + n_1 = 2n$, has the unique solution $(n, 0)$. The last equation from above, that is $2n_0 + n_1 = 2n - NOWL_3 + 1$, has multiple solutions, that are $(n - \frac{NOWL_3 - 1}{2}, 0), \ldots, (n - NOWL_3 + 1, NOWL_3 - 1)$. Let Δ_2 be the triangle with vertices $(n, 0)$, $(n - \frac{NOWL_3 - 1}{2}, 0)$ and $(n - NOWL_3 + 1, NOWL_3 - 1)$, that is, the triangle given by the n_0-axis, the group size constraint and the NOPA approximation constraint, where the latter is derived from the last equation of the set (52). This triangle contains as lattice points all solutions of the system of equations, which is shown above.

In this context, it is worth noting that $(n - \frac{NOWL_3 - 1}{2}, 0)$ may not be a lattice point and, therefore, we have to distinguish two cases, which are: $n - \frac{NOWL_3 - 1}{2} \in \mathbb{N}$ and $n - \frac{NOWL_3 - 1}{2} \notin \mathbb{N}$. In the second case, we need to add an auxiliary line to get a lattice polygon which allows us to use Pick's formula (48). In general, the auxiliary line simply links the following two lattice points $(\lceil n - \frac{NOWL_3 - 1}{2} \rceil, 0)$ and $(\lceil n - \frac{NOWL_3 - 1}{2} \rceil - 1, 1)$. The lattice polygon contains the same number of lattice points as Δ_2, but, in the case of Figure 1, differs by $-1/4$ in the size of the area. Thus, we derive (see appendix),

$$NOPA \leq \left\lfloor \frac{(NOWL_3 - 1)^2}{4} \right\rfloor + NOWL_3. \tag{53}$$

[12] Actually, each boundary edge has $n + 1$ lattice points, but since the three lattice points situated at the vertices (corners) of the triangle are counted twice, we get $3(n + 1) - 3 = 3n$.

Hence, the first approximation of the upper bound of $NOPA$ amounts to the sum of the number of lattice points of the triangle with vertices $(1.5, 0)$, $(5, 0)$ and $(-2, 7)$. Note that this is an example for Δ_2, and these lattice points are marked by crosses, boxes and diamonds, which sum up to 20. In addition, it is also an example for the second case and, therefore, an auxiliary line is added in Figure 1, which yields the lattice polygon with vertices $(1, 1)$, $(2, 0)$, $(5, 0)$ and $(-2, 7)$. For $n = 5$, the three lattice points marked by crosses are in the negative (second quadrant) and, therefore, do not belong to the set of feasible allocations. Thus, we get $NOPA \leq 20 - 3 = 17$, which coincides with the result shown in the preceding section, and is visualized in Figure 1 by lattice points marked by boxes and diamonds.

To proceed, we now calculate $NOPA$. With respect to (22) and (52), we have the following constraint for Pareto-optimal allocations,

$$n_0 \geq n_{min} = \left\lceil n - \frac{NOWL_3 - 1}{2} \right\rceil. \tag{54}$$

Applying (54) or inspection of Table 1 yields $n_{min} = 2$, which is denoted by the dotted vertical line in Figure 1. To calculate $NOPA$ we need to sum up lattice points of the triangle Δ_3, with vertices $(n, 0)$, $(n_{min}, 0)$ and (n_{min}, n_{max}), that is, the triangle given by the n_0-axis, the groups size constraint and the n_{min}-line. In Figure 1 these vertices amount to (5,0), (2,0) and (2,3), respectively. Using (48) in the same manner as above yields $F = \frac{1}{2}n_{max}^2$ and $\hat{G} = 3n_{max}$. Thus, in accordance with (46) we obtain,

$$NOPA = \frac{1}{2}n_{max}^2 + \frac{3}{2}n_{max} + 1 = \frac{(n_{max} + 2)!}{n_{max}!2!} = \binom{n_{max} + 2}{n_{max}}. \tag{55}$$

Application of (55) yields $NOPA = 10$, which in Figure 1 is represented by lattice points marked by diamonds.

Figure 1 is restricted to cases where the set of alternatives is equal to three, that is, $m = 3$, $\forall i \in \Im_n$. Of course, it can be applied to $m = 2$, $\forall i \in \Im_n$ as well, but then Figure 1 degenerates to one dimension only, represented by the n_0-axis, on which the six allocations of Table 2 appear. Moreover, extensions of Pick's formula are available that may be used for those cases where the set of alternatives is larger than three. De Loera et al. (2004) provide such an extension and also offer a generalized calculation code that can be applied to these cases. The code can be downloaded at: http://www.math.ucdavis.edu/~latte. In particular, the code can be used to independently verify the calculation results obtained from our own code, which is provided in the appendix. Note, however, that the De Loera et al. code runs under Linux and may require a somewhat more complex input than the code we provide in this paper.

3.4 Second Scenario: General Case

We now consider the second scenario, where agents are uniquely distinguishable. In particular, we make the same assumptions as in the first scenario, except that the normal form of the linear public goods game now is $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B;m;1), \ldots, (\alpha;\beta;\gamma;\delta;B;m;1)\}$, according to (14), where each tuple describes individual characteristics of the n agents. We continue to regard W as an onto correspondence and obtain two finite sets, the domain $\mathfrak{D} := \{(x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{N}_0^n \mid \sum_{i=1}^n x_i = X\}$, containing all feasible contributions to the public good $x_1, x_2, \ldots, x_{n-1}, x_n$, and the image $\mathfrak{W} := \{\alpha\zeta(m-1)n, \alpha\zeta((m-1)n-1)+\beta\eta n, \ldots, \alpha\zeta+\beta\eta((m-1)n-1)n, \beta\eta(m-1)n^2\}$, containing all feasible levels of welfare. Note that the domain $\mathfrak{D}$ differs from the domain used in section 3.2, whereas the image $\mathfrak{W}$ is identical in both cases.

Again, applying (9) requires calculating how many elements $\mathfrak{D}$ has, and applying (10) requires calculating how many elements $\mathfrak{W}$ has. Using combinatorics[13] we obtain,

$$NOA = |\mathfrak{D}| = m^n \tag{56}$$

and,

$$NOWL = |\mathfrak{W}| = (m-1)n + 1. \tag{57}$$

Hence, as noted, for the numerical case of section 2.2 we get $NOA = 3^5 = 243$ and $NOWL = 2 \cdot 5 + 1 = 11$.

To proceed, it is important to keep in mind that the necessary and sufficient condition for calculating $NOPA$ in the first scenario, $n_0 \geq n_{min}$, according to (22), continues to hold. This is because the condition is independent from whether agents can be uniquely distinguished or not. Therefore, we can apply a similar calculation procedure for $NOPA$, that is, we calculate the number of allocations with k free-riders by using combinatorics[14] and get,

$$NOA(k) = \sum_{\sum_{j=1}^{m-1} n_j = k} \frac{n!}{(n-k)! \prod_{j=1}^{m-1} n_j!}, \qquad \forall k \in \{0, 1, \ldots, n\}. \tag{58}$$

For the numerical case of section 2.2 we obtain,

$NOA(0) = \sum_{n_1+n_2=0} \frac{5!}{(5-0)!n_1!n_2!} = 1$

$NOA(1) = \sum_{n_1+n_2=1} \frac{5!}{(5-1)!n_1!n_2!} = 5 + 5 = 10$

$NOA(2) = \sum_{n_1+n_2=2} \frac{5!}{(5-2)!n_1!n_2!} = \frac{5!}{3!2!0!} + \frac{5!}{3!1!1!} + \frac{5!}{3!0!2!} = 10 + 20 + 10 = 40$

$NOA(3) = \sum_{n_1+n_2=3} \frac{5!}{(5-3)!n_1!n_2!} = \frac{5!}{2!3!0!} + \frac{5!}{2!2!1!} + \frac{5!}{2!1!2!} + \frac{5!}{2!0!3!} = 10 + 30 + 30 + 10 = 80$

which yields $NOPA = \sum_{k=0}^{3} NOA(k) = 131$ (see Table 1). In general, we get,

$$NOPA = \sum_{k=0}^{n_{max}} NOA(k) = \sum_{k=0}^{n_{max}} \sum_{\sum_{j=1}^{m-1} n_j = k} \frac{n!}{(n-k)! \prod_{j=1}^{m-1} n_j!}, \tag{59}$$

where n_{max} is calculated as in the first scenario.

We now turn to visualizing NOA, $NOWL$ and $NOPA$ with respect to the second scenario. To do so, we add a column for the number of clone allocations (CA), as shown in Table 1 and Table 2. A clone allocation indicates that these allocations are not Pareto-distinguishable from their associated master allocation. For $n_0, n_1, \ldots, n_{m-1}$ fixed, the master allocation is simply an arbitrarily chosen allocation from the set of $n!/\prod_{j=0}^{m-1} n_j!$-feasible allocations due to the permissible permutations of agents. Hence, the number of master allocations coincide with the number of allocations in the first scenario.

[13] Regarding NOA, each element of the n elements (agents) in the domain (i.e. $x_1, x_2, \ldots, x_n$) has m alternatives to occur, so that we get m^n (see Read 1972, 128, Example 3).

[14] Equation (58) is derived in two steps. First, following Read (1972, 108) we consider an allocation of the first scenario and calculate permutations by using the general formula for the number of permutations of n objects, $(n-k)$ of one type (full contributors), n_1 of another type (say partial contributors), and so on until we get to the n_{m-1} type (non-contributors). Second, to obtain $NOA(k)$, the numbers of permutations for each allocation with k free-riders must be added up.

In Table 1, we obtain NOA of the second scenario by adding-up the number of clone allocations, which yields 222, plus the 21 master allocations, so that we get $222 + 21 = 243$. In contrast, $NOWL$ is identical in both scenarios and amounts to 11. Finally, $NOPA$ is obtained by adding up the clone allocations of the Pareto-optimal master allocations, which yields 121, plus the 10 Pareto-optimal master allocations, so that we get $121 + 10 = 131$. Thus, the Pareto-ratio is $131/243 \approx 0.539$.

Results of this subsection may be summarized as follows.

Result 4: *Let the parameters n, α, β, γ, and δ be selected such that condition (7) holds, so that a prisoner's dilemma prevails, and let the normal form of the linear public goods game be $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B;m;1),\ldots,(\alpha;\beta;\gamma;\delta;B;m;1)\}$, according to (14). Then, the number of allocations (NOA) is calculated from (56), which is*

$$NOA = m^n,$$

the number of welfare levels (NOWL) is calculated from (57), which is

$$NOWL = (m-1)n + 1,$$

and the number of Pareto-optimal allocations (NOPA) is calculated from (59), which is

$$NOPA = \sum_{k=0}^{n_{max}} \sum_{\sum_{j=1}^{m-1} n_j = k} \frac{n!}{(n-k)! \prod_{j=1}^{m-1} n_j!},$$

where the maximum number of free-riders (n_{max}) just tolerated by the Pareto-optimality concept is calculated from (23), which is $\left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil - 1$.

3.5 Necessary and Sufficient Conditions

The relevant syntax to prove necessary and sufficient conditions with respect to Pareto-optimality and prisoner dilemma situations is now available and, therefore, we proceed with providing these proofs.

Proposition 1
In a standard linear public goods game a prisoner's dilemma situation arises if (7), including (6), holds. That is,

$$\frac{1}{n} < \frac{\beta\gamma}{\alpha\delta} < 1.$$

Proof 1 (necessary and sufficient condition for prisoner dilemma situations)

For a prisoner's dilemma situation to arise, the necessary condition is that for each agent the incremental return from investing an additional ϵ unit into the private good must exceed the incremental return from investing that ϵ unit into the public good, that is, $\alpha\zeta > \beta\eta$, according to the payoff function (2). If the necessary condition is fulfilled, at least one Nash equilibrium exists, which, for each agent, is non-contribution to the public good. Rearranging and using the production constraint (5) yields $1 < \frac{\alpha\delta}{\beta\gamma}$. In addition, the sufficient condition for a prisoner's dilemma situation is a positive group benefit of an incremental contribution to the public good, which according to (31) amounts to $\beta\eta n - \alpha\zeta > 0$. The sufficient condition ensures that at least full contribution (i.e., contributing the entire budget) to the public good is Pareto-optimal. Again, rearranging and using the production constraint (5) yields $\frac{\alpha\delta}{\beta\gamma} < n$. Thus, we obtain $1 < \frac{\alpha\delta}{\beta\gamma} < n$. Taking reciprocals yields the necessary and sufficient condition according to (7). ■

Proposition 2
In a standard linear public goods game the necessary condition for Pareto-optimal allocations is according to (37) and (38),

$$X > X_{min} = (\eta\, n - \frac{\alpha}{\beta}\, \zeta)(m-1).$$

Proof 2 (necessary condition for Pareto-optimal allocations)

If the necessary and sufficient condition (7) is fulfilled, the allocation associated with the benchmark according to (34), is Pareto-optimal. In addition, to prove by contradiction, suppose that $X \leq X_{min}$ holds. Then, the individual payoff of a non-contributor is $\alpha\zeta(m-1) + \beta X \leq \alpha\zeta(m-1) + \beta X_{min} = \alpha\zeta(m-1) + \beta(\eta\, n - \frac{\alpha}{\beta}\, \zeta)(m-1) = \beta\eta\, (m-1)n$. The term on the right hand side is the individual payoff that represents the benchmark. Thus, an allocation with $X \leq X_{min}$ cannot be Pareto-optimal, because there exists at least one allocation, the benchmark allocation, that represents a Pareto-improvement. Note that on purely formal grounds the necessary condition also includes the benchmark allocation, because in this case there is no deviating agent who needs to get a higher individual payoff. Therefore, as stated above, $X > X_{min}$, is the necessary condition for Pareto-optimal allocations according to (38). ■

Proposition 3
In a standard linear public goods game the sufficient condition for Pareto-optimal allocations is according to (22) and (44),

$$n_0 \geq n_{min} = n - \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil + 1,$$

which states that the number of full contributors must be larger than or equal to the minimum number of full contributors required for Pareto-optimality.

Proof 3 (sufficient condition for Pareto-optimal allocations)

Suppose a typical linear public goods game with a homogeneous parameter setting prevails. Then, an algorithm to obtain allocations where agents get the same or a higher individual payoff may be constructed.[15] Consider an initial allocation where $n_0 < n_{min}$ holds, i.e. an allocation that is not Pareto-optimal.[16] We denote the quantity of the public good in this allocation as $X_{(0)} = \sum_{j=0}^{m-1} \eta n_j(m-j-1)$. Then, according to (28), (29), and (30) the individual payoff of agents on alternative PC_k is $U_k = \alpha\zeta k + \beta \sum_{j=0}^{m-1} \eta n_j(m-j-1)$, $k \in \mathfrak{I}_m$. In each step, the algorithm requires full contributors to continue to contribute their entire budget to the public good and requires partial contributors and non-contributors to contribute an additional ϵ unit to

[15]This algorithm is developed here simply for the purpose of Proof 3. Due to the prevailing prisoner's dilemma it may not work in an experiment with human subjects.
[16]Note that the algorithm also applies to cases where $n_0 \geq n_{min}$ holds.

the public good. In this case, in each step of the algorithm, full contributors get a higher payoff, whereas partial contributors (including former non-contributors) may get a higher payoff, but at least they get the same payoff as before. Therefore, the resulting allocation cannot represent a Pareto-deterioration, because no agent is worse off. Moreover, if there are full contributors, each full contributor is necessarily better off and, thus, the new allocation must represent a Pareto-improvement. This is because the quantity of the public good in the new allocation (obtained by step one of the algorithm), that is, $X_{(1)} = \sum_{j=1}^{m-1} \eta n_j(m-j) + \eta n_0(m-1)$, leads to a payoff increase of $\beta(X_{(1)} - X_{(0)}) = \beta\eta(n - n_0)$, which represents the payoff increase for each full contributor. Furthermore, all agents who have contributed an additional ϵ unit incur an individual loss of $\alpha\zeta$, but gain $\beta\eta(n-n_0)$ from the public good, so that $\beta\eta(n-n_0) \geq \alpha\zeta$ prevails for each of these agents. Rearranging and using the production constraint (5) then yields $n_0 < n - \frac{\alpha\delta}{\beta\gamma} + 1$, which ensures a Pareto-improvement. The latter is always guaranteed, but two cases have to be distinguished. First, if there is no full contributor in the allocation under consideration ($n_0 = 0$), all agents are necessarily better off because the sufficient condition for a prisoner's dilemma situation must hold, that is, $\beta\eta n > \alpha\zeta$ (see Proof 1). Second, if there is at least one full contributor in the allocation under consideration ($n_0 > 0$), for reasons mentioned above, this full contributor is necessarily better off. Note, however, that the new allocation may not yet belong to the set of Pareto-optimal allocations.

Next, it is convenient to re-write the new allocation that results from applying step one of the algorithm. We denote the new number of full contributors as $N_{(0,1)} := n_0 + n_1$, where the first index indicates the alternative and the second index indicates the step of the algorithm. Likewise, the new number of partial contributors is denoted by $N_{(k,1)} := n_{k+1}$, $k \in \{1, \ldots, m-2\} \subset \mathfrak{I}_m$. Note that after the first step of the algorithm the new number of non-contributors necessarily equals zero, because every initial non-contributor has contributed at least one ϵ unit to the public good. Formally we denote this by $N_{(m-1,1)} := 0$. With respect to the r-th step of the algorithm, that is, r-times applying the first step, $r \in \mathbb{N}$, we denote the number of full contributors by $N_{(0,r)} := \sum_{j=0}^{r} n_j$ and the number of partial contributors as $N_{(k,r)} := n_{k+r}$, $k \in \{1, \ldots, m-r-1\}$, $\forall r \in \{1, \ldots, m-1\}$. Consequently, in this case there are no agents on alternatives PC_{m-r}, $\ldots$, PC_{m-1}, which we denote as $N_{(k,r)} := 0$, $k \in \{m-r, \ldots, m-1\}$, $\forall r \in \{1, \ldots, m-1\}$, respectively. Given these definitions, after r-steps the quantity of the public good amounts to $X_{(r)} = \sum_{j=0}^{m-1} \eta N_{(j,r)}(m-j-1)$.

Since by definition we have $n_{min} < n$, it follows that after a finite number of $S \in \mathbb{N}$ steps, where $S < m$ holds, we must necessarily have $\sum_{j=0}^{S} n_j \geq n_{min}$. Therefore, the algorithm is applied in S steps until the truncation constraint for Pareto-optimality is fulfilled, that is, $N_{(0,S)} \geq n_{min}$. Put differently, after S steps the prevailing allocation must be Pareto-optimal.

Now assume that such an allocation prevails.[17] If the algorithm is applied once more, so that each free-rider (non-full contributor) contributes an additional ϵ unit, whereas full contributors continue contributing their entire budget to the public good, the new allocation (obtained by step $S+1$ of the algorithm) must necessarily represent a Pareto-deterioration. A proof of the latter statement is given by contradiction as follows.

The quantity of the public good provided after S-steps is $X_{(S)} = \sum_{j=0}^{m-1} \eta N_{(j,S)}\,(m-j-1)$ and after $S+1$-steps it amounts to $X_{(S+1)} = \sum_{j=1}^{m-1} \eta N_{(j,S)}\,(m-j) + \eta N_{(0,S)}\,(m-1)$.

[17] Since the application of the algorithm to the benchmark allocation obviously generates the benchmark allocation again, and the latter is Pareto-optimal anyway, we exclude the benchmark allocation from this framework.

Hence, compared with step S applying the algorithm for $S+1$-steps leads to an individual payoff increase of $\beta(X_{(S+1)} - X_{(S)}) = \beta\eta(n - N_{(0,S)})$ and $N_{(0,S)} \geq n_{min}$ yields $\beta\eta(n - N_{(0,S)}) \leq \beta\eta(n - n + \frac{\alpha\zeta}{\beta\eta} - 1) = \alpha\zeta - \beta\eta$. Again, each additional contributing agent has an individual loss of $\alpha\zeta$. In contradiction to Pareto-optimality these agents incur an individual net loss of at least $\beta\eta$ and, therefore, the new allocation must represent a Pareto-deterioration.

In summarizing, the algorithm generates a Pareto-improvement until the truncation constraint is fulfilled, but if applied thereafter it leads to a Pareto-deterioration. Hence, if the truncation condition holds and one or more agents continue to contribute one or more additional ϵ units to the public good, because of $N_{(0,S)} \geq n_{min}$, at least one of these agents necessarily incurs a net individual loss. This is because the group of free-riders (non-full contributors) is too small to generate an individual payoff increase that exceeds for each additional contributing agent the individual loss. Thus, we have shown that under these circumstances there exists no option in the set of feasible allocations that makes at least one agent better off in terms of its own individual payoff and, at the same time, none of the other agents worse off, which happens to be the definition of Pareto-optimality. ■

Having solved the calculation problem for NOA, $NOWL$, $NOPA$, and the Pareto-ratio for homogeneous parameter settings in both the first and second scenario, we proceed with heterogeneous parameter settings in the next section.

4 Tracing Pareto-optimality: Heterogeneous Settings

In this section we further generalize the calculation procedure by considering linear public goods games where agents either have heterogeneous incomes (budgets) or heterogeneous $MPCR$s (productivities).

4.1 Individualized Linear Public Goods Games

We continue with all model specifications made in the preceding section, except that all relevant functions and parameters may now be fully individualized, with B_i, U_i, m_i, α_i, β_i, γ_i, δ_i, ϵ_i, ζ_i, and η_i, $\forall i \in \Im_n$. Hence, in this framework the i-th agent is completely characterized by a tuple $(\alpha_i;\beta_i;\gamma_i;\delta_i;B_i;m_i)$. Let $P \in \Im_n$ be the number of different tuples among all tuples that actually appear. Let $f^p \in \Im_n$ be the number of agents endowed with the same tuple $(\alpha^p;\ \beta^p;\ \gamma^p;\ \delta^p;\ B^p;\ m^p)$, $\forall p \in \{1,\ldots,P\}$, which are indistinguishable if their individual contribution to the public good is identical.[18] Note that it is essential to just allow for tuples that actually appear, i.e. $f^p > 0, \forall p \in \{1,\ldots,P\}$. Put differently, we consider f^p as the number of agents in subgroup p, where no subgroup may be empty. By definition we get,

$$\sum_{p=1}^{P} f^p = n, \tag{60}$$

and, therefore, given our specifications the generalized normal form of the linear public goods game $\mathfrak{F}$ is,

[18] A superscript is used to stress the difference between individual tuples $i \in \Im_n$ and different tuples $p \in \{1,\ldots,P\}, P \in \Im_n$.

$$\mathfrak{F} := \{(\alpha^p; \beta^p; \gamma^p; \delta^p; B^p; m^p; f^p) \mid p \in \{1, \ldots, P\}\}, \tag{61}$$

where homogeneous parameter settings of the preceding section (see(13) and (14)) are covered by either $P = 1$, $f^1 = n$ or $P = n$, $f^p = 1$, $\forall p \in \{1, \ldots, n\}$, respectively.

Moreover, we infer from the individualized versions of (1), (2) and (3) that the i-th agent's individual marginal per capita return $MPCR_i$ of a contribution to the public good, is,

$$MPCR_i = \frac{\beta_i \gamma_i}{\alpha_i \delta_i} \qquad \forall i \in \mathfrak{I}_n, \tag{62}$$

which extends the earlier definition of $MPCR$. For convenience we generalize (8), which is the sum of individual payoffs,

$$W(x_1, \ldots, x_n) := \sum_{i=1}^{n} U_i(y_i, X). \tag{63}$$

It is important to recognize that (63) depends on n variables and not on $n + 1$ variables. This is because y_i is automatically given by applying (3), if the entire budget B_i is spend, and X emerges from summing up the n individual contributions to the public good, i.e., from (1).

Furthermore, for each agent of subgroup $p \in \{1, \ldots, P\}$ the individualized necessary condition for a prisoner's dilemma situation is,

$$\frac{\beta^p \gamma^p}{\alpha^p \delta^p} < 1, \tag{64}$$

where section 3.5, Proof 1, applies in an analogous manner with respect to the necessary condition. Regarding the sufficient condition the individual payoff in the benchmark allocation (each agent fully contributes to the public good) has to be greater than in Nash equilibrium (each agent does not contribute to the public good), because otherwise the benchmark allocation would not represent a Pareto-improvement. Formally, for each agent of subgroup $p \in \{1, \ldots, P\}$ we obtain as individualized sufficient condition,

$$\alpha^p \zeta^p (m^p - 1) < \beta^p \sum_{i=1}^{n} \eta_i (m_i - 1). \tag{65}$$

Note, however, that in a group of size n, for some subgroups p of size f^p neither the necessary nor the sufficient condition may hold, while at the same time both conditions may hold for other subgroups. Hence, there may be groups where for some agents a prisoner's dilemma holds, while simultaneously it does not hold for other agents in that same group. Such a situation may allow for interesting new experimental designs, as we demonstrate in the following subsections.

4.2 Heterogeneous Endowment Settings

We now assume that each agent is endowed with an individual budget $B_i \in \mathbb{N}, \forall i \in \mathfrak{I}_n$, which may differ among agents. Otherwise, however, we continue to assume a homogeneous parameter setting. Then, provided that ϵ is kept constant, it follows from the generalized functional form (4) that each agent may have an individual number of alternatives $|\mathfrak{A}_i| = m_i \geq 2$, $\forall i \in \mathfrak{I}_n$ and, therefore, the i-th agent is characterized in the linear public goods game by the tuple $(\alpha; \beta; \gamma; \delta; B_i; m_i)$. Moreover, according to (61) linear public goods games with heterogeneous endowment settings are characterized by $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B^p;m^p;f^p) \mid p \in \{1,\ldots,P\}\}$.

Again, by using combinatorics[19] we obtain,

$$NOA = \prod_{p=1}^{P} \binom{f^p + m^p - 1}{f^p} = \prod_{p=1}^{P} \frac{(f^p + m^p - 1)!}{f^p!(m^p - 1)!} \tag{66}$$

and

$$NOWL = \sum_{p=1}^{P} f^p m^p - n + 1, \tag{67}$$

which are generalizations of (18), (32), (33), (56), and (57).

It is worth noting that (64) holds for each subgroup and the calculation procedures for NOA and $NOWL$ do not change, if agents in some subgroups do not face a prisoner's dilemma situation according to (65). Moreover, homogeneous parameter settings are included in this framework by the condition that each agent is endowed with the same constant budget. For example, consider the numerical example given in section 2.2, where the parameter set of the first scenario is $\alpha = 4$, $\beta = \gamma = \delta = \epsilon = \zeta = \eta = 1$, $P = 1$, $B^1 = 2$, $m^1 = 3$ and $f^1 = 5$ yields $\mathfrak{F} = \{(4;1;1;1;2;3;5)\}$, applying (66) and (67) yields $NOA = 21$ and $NOWL = 11$, respectively. Likewise, with respect to the second scenario, where we get $P = 5$ and $f^p = 1$, $\forall p \in \{1,2,3,4,5\}$, which amounts to $\mathfrak{F} = \{(4;1;1;1;2;3;1), \forall p \in \{1,2,3,4,5\}\}$, applying (66) and (67) yields $NOA = 3^5 = 243$ and $NOWL = 11$, respectively (see Table 1).

4.2.1 NOPA Calculation Procedure

Regarding $NOPA$ we use the algorithm of section 3.5, Proof 3. Applying (4) we derive the number of alternatives $m^{p^*} \in \mathbb{N}$ used in the algorithm, where m^{p^*} refers to the subgroup $p^* \in \{1,\ldots,P\}$ endowed with the largest budget $B^{p^*} \geq B^p$, $\forall p \in \{1,\ldots,P\}$. Then, these richest agents may choose from the set of alternatives $\mathfrak{A}^{p^*} = \{PC_0, PC_1, \ldots, PC_{m^{p^*}-1}\}$, whereas agents of all other subgroups may only choose from a subset of these alternatives $\mathfrak{A}^p = \{PC_0, PC_1, \ldots, PC_{m^p-1}\} \subseteq \mathfrak{A}^{p^*}$. Yet, the truncation condition of the algorithm is not influenced by income heterogeneity and continues to hold. Put differently, $n_0 \geq n - \left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil + 1$ is the necessary and sufficient condition for Pareto-optimality in heterogeneous endowment settings. Applying (23) yields n_{max} and, therefore, $NOPA$ is given by,

$$NOPA = \sum_{k=0}^{n_{max}} NOA(k). \tag{68}$$

Hence, (68) is a generalization of (25), (46), and (59), but because of the variety of cases that may occur, we are unable to determine the number of allocations with k free-riders (non-full contributors).

To summarize, we are left with the following result,

Result 5: *Let the parameters n, α, β, γ, and δ be selected such that a prisoner's dilemma prevails for agents of at least one subgroup $p \in \{1,\ldots,P\}$, let the normal*

[19] Essentially, we apply again the formula given by Read (1972, 111) for each subgroup p and then multiply the results.

form of the linear public goods game be $\mathfrak{F} = \{(\alpha;\beta;\gamma;\delta;B^p;m^p;f^p) \mid p \in \{1,\ldots,P\}\}$*, according to (61), and assume that the constraints* $\epsilon_i = \epsilon \leq B_i$*,* $\zeta_i = \zeta$*,* $\eta_i = \eta \in \mathbb{N}$*,* $\forall i \in \mathfrak{I}_n$ *are fulfilled. Then, the number of allocations (NOA) is calculated from (66), which is*

$$NOA = \prod_{p=1}^{P} \frac{(f^p + m^p - 1)!}{f^p!(m^p - 1)!},$$

the number of welfare levels (NOWL) is calculated from (67), which is

$$NOWL = \sum_{p=1}^{P} f^p m^p - n + 1,$$

and the number of Pareto-optimal allocations (NOPA) is calculated from (68), which is

$$NOPA = \sum_{k=0}^{n_{max}} NOA(k),$$

where the maximum number of free-riders (n_{max}) *just tolerated by the Pareto-optimality concept is calculated from (23), which is* $\left\lceil \frac{\alpha\delta}{\beta\gamma} \right\rceil - 1$.

4.3 Heterogeneous MPCR Settings

To examine heterogeneous $MPCR$s we consider individualized payoff function parameters, α_i and $\beta_i \in \mathbb{Q}^+$, and individualized budget constraint parameters, γ_i and $\delta_i \in \mathbb{Q}^+$, $\forall i \in \mathfrak{I}_n$. In addition, we assume a homogeneous ϵ unit, which subject to (5) yields individualized production constraints ζ_i and $\eta_i \in \mathbb{N}$, $\forall i \in \mathfrak{I}_n$. Otherwise, we continue to assume a homogeneous parameter setting. Therefore, the i-th agent is characterized in the game by the tuple $(\alpha_i;\beta_i;\gamma_i;\delta_i;B;m)$. According to (61) linear public goods games with heterogeneous $MPCR$ settings are characterized by $\mathfrak{F} = \{(\alpha^p;\beta^p;\gamma^p;\delta^p;B;m;f^p) \mid p \in \{1,\ldots,P\}\}$.

Again, by using combinatorics we obtain,

$$NOA = \prod_{p=1}^{P} \binom{f^p + m - 1}{f^p} = \prod_{p=1}^{P} \frac{(f^p + m - 1)!}{f^p!(m - 1)!}, \tag{69}$$

which simplifies (66). Note that (69) continues to hold, even if agents in some subgroups do not face a prisoner's dilemma situation.

Further, with respect to $NOWL$ we need to examine the onto correspondence W according to (63). To give a simple generalized example, we now assume identical payoff functions and that each agent is faced with a prisoner's dilemma situation according to (64) and (65). If no agent contributes to the public good, that is, NC ($y_i = \zeta_i(m-1)$, $x_i = 0$), $\forall i \in \mathfrak{I}_n$, the Nash equilibrium and, therefore, the minimum level of welfare occurs, that is, $W_{min} = \alpha \sum_{i=1}^{n} y_i = \alpha(m-1) \sum_{i=1}^{n} \zeta_i = \alpha(m-1) \sum_{p=1}^{P} f^p \zeta^p$. In contrast, if all agents contribute their entire budget to the public good, that is, FC ($y_i = 0$, $x_i = \eta_i(m-1)$), $\forall i \in \mathfrak{I}_n$, the maximum level of welfare occurs and amounts to, $W_{max} = \beta n \sum_{i=1}^{n} x_i = \beta(m-1)n \sum_{i=1}^{n} \eta_i = \beta(m-1)n \sum_{p=1}^{P} f^p \eta^p$. We define the minimum of the smallest possible unit in which the quantity of the private good may be produced as,

$$\zeta_{min} := \min_{p \in \{1,\ldots,P\}} \{\zeta^p\}, \tag{70}$$

and, in an analog manner, the minimum quantity of the public good as,

$$\eta_{min} := \min_{p \in \{1,\ldots,P\}} \{\eta^p\}. \tag{71}$$

Then, the incremental group payoff of a minimum contribution to the public good is at least $\beta\eta_{min}n - \alpha\zeta_{min}$. Using this information, we obtain a lower and upper bound for $NOWL$, which is,

$$(m-1)n + 1 \leq NOWL \leq \frac{W_{max} - W_{min}}{\beta\eta_{min}n - \alpha\zeta_{min}} + 1. \tag{72}$$

Moreover, note that in homogeneous parameter settings the right hand side of (72) is given by $\frac{\beta\eta(m-1)n^2 - \alpha\zeta(m-1)n}{\beta\eta n - \alpha\zeta} + 1$ and, therefore, degenerates to $(m-1)n + 1$, which yields (33) and (57).

To generalize, in heterogeneous $MPCR$ settings we need to specify the onto correspondence W with a view to yield a numerical solution. A spreadsheet may then be used to count the number of welfare levels ($NOWL$).

4.3.1 NOPA Calculation Procedure

We use the minimum quantity of the public good that must necessarily be exceeded to achieve Pareto-optimality (here X^p_{min}) and apply similar calculation procedures as in sections 3.1 and 3.2, which yields for each agent an individualized benchmark,

$$\widehat{U}_{i,0} := \beta_i \sum_{p=1}^{P} \eta^p (m-1) f^p, \qquad \forall i \in \Im_n, \tag{73}$$

where the first index represents the agent, $i \in \Im_n$, and the second index indicates the alternative (full contribution). The necessary condition for Pareto-optimal allocations is that non-contributors gain a higher individual payoff than the benchmark, which amounts to,

$$\alpha_i(m-1)\zeta_i + \beta_i X > \beta_i \sum_{p=1}^{P} \eta^p (m-1) f^p, \qquad \forall i \in \Im_n, \tag{74}$$

and, therefore, by rearranging,

$$X > (-\frac{\alpha_i\zeta_i}{\beta_i} + \sum_{p=1}^{P} \eta^p f^p)(m-1), \qquad \forall i \in \Im_n. \tag{75}$$

Note that homogeneous parameter settings are included in this framework and $\eta_i = \eta$, $\forall i \in \Im_n$, yields $\sum_{p=1}^{P} f^p\eta^p = \eta n$, so that (75) coincides with (36). To proceed, we now define for subgroup $p \in \{1, \ldots, P\}$ the quantity of the public good that needs to be exceeded in order to achieve Pareto-optimality as,

$$X^p_{min} := (-\frac{\alpha^p\zeta^p}{\beta^p} + \sum_{p=1}^{P} \eta^p f^p)(m-1). \tag{76}$$

Further, X^p_{min} may be used to check whether or not Pareto-optimality of an allocation can be ruled out. If the following condition holds for all subgroups, the benchmark allocation must represent a Pareto-improvement, and, therefore, the allocation cannot be Pareto-optimal,

$$X \leq X^{p}_{min}, \qquad \forall p \in \{1, \ldots, P\}. \tag{77}$$

To obtain the sufficient condition for Pareto-optimality, we now slightly modify the algorithm of section 3.5, Proof 3. Consider an arbitrarily selected, non-Pareto-optimal allocation from the set of feasible allocations. We call this allocation the 'initial allocation' and denote the quantity of the public good provided by subgroup $p \in \{1, \ldots, P\}$ as $X^{p}_{(0)} = \sum_{j=0}^{m-1} \eta^{p} f^{p}_{j}(m - j - 1)$, where f^{p}_{j} is the number agents of subgroup p who have selected alternative PC_j, with $j \in \mathfrak{I}_m$. The public good provided by the group of n agents in the initial allocation amounts to $X_{(0)} = \sum_{p=1}^{P} X^{p}_{(0)} = \sum_{p=1}^{P} \sum_{j=0}^{m-1} \eta^{p} f^{p}_{j}(m-j-1)$. The algorithm requires, that each full contributor continues to devote the entire budget to the public good, whereas all partial contributors and non-contributors contribute one additional ϵ unit to the public good. Then, after the first step of the algorithm, the quantity of the public good amounts to $X_{(1)} = \sum_{p=1}^{P} X^{p}_{(1)} = \sum_{p=1}^{P} \sum_{j=1}^{m-1} \eta^{p} f^{p}_{j}(m - j) + \eta^{p} f^{p}_{0}(m - 1)$, where the additionally provided quantity of the public good is $X_{(1)} - X_{(0)} = \sum_{p=1}^{P} \sum_{j=1}^{m-1} \eta^{p} f^{p}_{j}$. Therefore, each agent has an individual payoff increase that amounts to $\beta^{p} \sum_{p=1}^{P} \sum_{j=1}^{m-1} \eta^{p} f^{p}_{j} = \beta^{p} \sum_{p=1}^{P} \eta^{p}(f^{p} - f^{p}_{0})$ and each additionally contributing agent incurs an individual loss of $\alpha^{p}\zeta^{p}$. Thus, $\beta^{p} \sum_{p=1}^{P} \eta^{p}(f^{p} - f^{p}_{0}) \geq \alpha^{p}\zeta^{p}$ ensures that the resulting allocation is no Pareto-deterioration. Note that this condition must hold for at least one subgroup, but may not hold for all subgroups. Therefore, the application of the algorithm is restricted to those subgroups for which the condition holds, because otherwise the application of the algorithm would make agents in all other subgroups worse off. However, if there is at least one full contributor in the initial allocation or the algorithm is repeatedly applied until there is at least one full contributor, the next application of the algorithm necessarily generates a Pareto-improvement. This is because any full contributor would gain from the increased quantity of the public good, but would not incur any additional individual loss due to contributing.

Hence, for the same reasons as in section 3.5, Proof 3, the algorithm eventually generates a Pareto-improvement until the truncation constraints,

$$\beta^{p} \sum_{p=1}^{P} \eta^{p}(f^{p} - f^{p}_{0}) < \alpha^{p}\zeta^{p}, \tag{78}$$

for all subgroups are fulfilled. Yet, if applied thereafter, that is, to a Pareto-optimal allocation, the algorithm necessarily leads to a Pareto-deterioration. However, due to the variety of cases that might occur, we refrain from displaying a general formula, but give some examples in section 5. Thus, in heterogeneous *MPCR* (productivity) settings we obtain the following result.

Result 6: *Let the parameters n, α_i, β_i, γ_i and δ_i, $\forall i \in \mathfrak{I}_n$, be selected such that a prisoner's dilemma prevails for agents of at least one subgroup $p \in \{1, \ldots, P\}$, let the normal form of the game be $\mathfrak{F} = \{(\alpha^{p}; \beta^{p}; \gamma^{p}; \delta^{p}; B; m; f^{p}) \mid p \in \{1, \ldots, P\}\}$, according to (61), and assume that the constraints $\epsilon_i = \epsilon \leq B$, ζ_i and $\eta_i \in \mathbb{N}$, $\forall i \in \mathfrak{I}_n$ are fulfilled. Then, the number of allocations (NOA) is calculated from (69), which is*

$$NOA = \prod_{p=1}^{P} \frac{(f^{p} + m - 1)!}{f^{p}!(m - 1)!},$$

the number of welfare levels (NOWL) may be counted by using a spreadsheet (and, provided each agent is faced with identical α, β and a prisoner's dilemma situation, may be approximated from (72), which is $(m-1)n+1 \leq NOWL \leq \frac{W_{max}-W_{min}}{\beta\eta_{min}n-\alpha\zeta_{min}}+1$, where the maximum and minimum level of welfare is $W_{max} = \beta(m-1)n\sum_{p=1}^{P} f^p\eta^p$ and $W_{min} = \alpha(m-1)\sum_{p=1}^{P} f^p\zeta^p$, respectively). Further, allocations where (77) holds, which is $X \leq X^p_{min}$, and X^p_{min} is $(-\frac{\alpha^p\zeta^p}{\beta^p}+\sum_{p=1}^{P} f^p\eta^p)(m-1)$ according to (76), are not Pareto-optimal. In contrast, allocations where the truncation constraints, $\beta^p\sum_{p=1}^{P}\eta^p(f^p - f^p_0) < \alpha^p\zeta^p$ according to (78), hold for all subgroups are Pareto-optimal.

Finally, to facilitate and simplify the application of our results we have developed a MATLAB code that is provided in the appendix. The code allows for reproducing virtually all tables and data shown in this paper, but may also be used for any other conceivable parameter setting and for up to five different income levels, subject to the general constraints mentioned so far. Therefore, the code may be applied to practically all linear public goods games and in the following section we demonstrate this with a selection of published linear public goods games.

5 Analysis of Published Linear Public Goods Games

We now apply the calculation procedure to various published linear public goods games. Table 3 provides an overview concerning the selected games and summarizes the results. The next subsection provides some general aspects concerning the selected games of Table 3. In the following subsections we use the calculation procedure for analyzing issues in linear public goods games, in particular, the actual occurrence of Pareto-optimal allocations, the decay of voluntary contributions, the contribution behavior of the poor and rich in a heterogeneous endowment setting, and the effect of redistributions on neutrality.

5.1 General Aspects

To set-up Table 3, in a first step, the relevant parameter values for B, m, n, α, β, γ, δ, ϵ, ζ, η and the $MPCR$ must be identified. These parameters are typically provided in the experimental design of these games, although γ, δ, ϵ, ζ, and η are often just implicitly given and m is calculated from (4). Inspection of Table 3 with respect to these parameter values shows that most experiments are conducted with ten or less subjects. In addition, most experimenters use an ϵ of one and just a few an ϵ smaller than one (i.e. Leuthold (1993), Weimann (1994) and Messer et al. (2007)) or larger than one (i.e. Brown-Kruse and Hummels (1993) and McCorkle and Watts (1996)). Further, all experimenters consider homogeneous budget endowments (incomes), except Cherry et al. (2005) and Buckley and Croson (2006), who consider heterogeneous endowments, and for Tan (2008), who considers heterogeneous $MPCR$s. Based on (4) and the choice of B and ϵ, subjects face a set of alternatives m that ranges from just two alternatives (McCorkle and Watts 1996) to 10,001 alternatives (Leuthold 1993), although in most cases subjects can select from a set of less than 100 alternatives. With respect to the parameters α, β, γ, δ, ζ, and η in all experiments they are homogeneous and γ, δ, ζ, and η are implicitly set equal to unity, except in Tan (2008), which we use to illustrate the case where agents face heterogeneous $MPCR$s.

Regarding the $MPCR$, an important observation in homogeneous and heterogeneous endowment settings is that whenever the $MPCR$ is 0.5 or higher, n_{max} is equal to one, and whenever the $MPCR$ is below 0.5, n_{max} is larger than one (see Table 3, and result three through five). Moreover, it follows that for any given set of α, β, γ and δ, n_{max} is fixed and any increase in n requires a corresponding increase in n_{min} according to (23). Put differently, if n increases *ceteris paribus*, the number of full contributors must increase by the same magnitude in order to preserve Pareto-optimality. Hence, the larger the group the more difficult it becomes to achieve a Pareto-optimal allocation. For example, in Table 3 this is demonstrated by Isaac et al. (1994) with respect to 4L, 10L, 40L and 100L. Then, if Pareto-optimality of an allocation is considered relevant for a group to voluntarily agree on providing a positive amount of public goods, smaller groups may find it easier to agree on providing public goods than larger groups. Of course, this would support Mancur Olson's (1965) claim that small groups will find it easier to agree on providing public goods than large groups.

5.2 Pareto-optimality

Identifying Pareto-optimal allocations would be a simple task if the actual number of full contributors (n_0 or n_{FC}) would be published for each round or trial, because in this case an allocation would be Pareto-optimal whenever $n_0 \geq n_{min}$ holds. Unfortunately, however, this is almost never done and Pareto-optimality must be inferred from the average contribution per round or trial, if possible.

To simplify the procedure, the last column on the right hand side of Table 3 indicates Pareto-optimality (PO), where '$-$' denotes that due to missing or incomplete results nothing can be said about Pareto-optimality, L denotes a low potential for Pareto-optimal allocations, H denotes a high potential for Pareto-optimal allocations, N denotes that there are no Pareto-optimal allocations and Y denotes that there are one or more Pareto-optimal allocations. We assume a high (low) potential for Pareto-optimal allocations if the actual average contribution in a round is above (below) the average contribution that results if a number of subjects equal to n_{min} fully contribute their budget, whereas all other subjects contribute nothing to the public good. Of course, this is just a rough measure and we cannot conclude whether or not an actual round was indeed Pareto-optimal, because the experimental results usually present an average over several sessions and the subjects can choose from many alternatives. Further, we claim that there are no Pareto-optimal allocations if the overall average contribution is below the percentage that would be needed for just one Pareto-optimal round. Finally, we claim Pareto-optimality for one or more allocations only if experimental results presented allow for a definite and positive conclusion.

As expected, Table 3 shows that Pareto-optimal allocations actually occur (Y), or may have a high potential to occur (H), predominantly in linear public goods games with an $MPCR$ below 0.5. However, a more detailed analysis of these cases reveals that often a contributions enhancing mechanism is employed. Therefore, in the next subsection we discuss these mechanisms in some detail.

5.2.1 Pareto-improving Mechanisms

In the following, we discuss all papers which either have a Y or H in column PO of Table 3, except those of Isaac and Walker (1988) and Isaac et al. (1994), because they do not apply a contributions enhancing mechanism and except Tan (2008), which we discuss in some detail in section 5.6.

Table 3: Analysis of Published Linear Public Goods Games

Author(s)	n	B	ϵ	m	α	β	$MPCR$	NOA	$NOWL$	$NOPA$	Pareto-ratio	n_{max}	n_{min}	PO
Andreoni(1995)	5	60	1	61	0.01	0.005	0.5	8,259,888	301	61	0.00000739	1	4	L
Andreoni and Petrie (2004)	5	20	1	21	0.02	0.01	0.5	53,130	101	21	0.00039526	1	4	L
Bardsley and Sausgruber (2005)	6	10	1	11	1	0.5	0.5	8,008	61	11	0.00137363	1	5	L
Bazart and Pickhardt (2011)	5	20	1	21	1	0.6	0.6	53,130	101	21	0.00039526	1	4	Y
Brandts and Schram (2001)														
Situation 3 Homo	4	9	1	10	15	12	0.8	715	37	10	0.01398601	1	3	L
	[4]	[9]	[1]	[10]	[15]	[12]	[0.8]	[10,000]	[37]	[37]	[0.0037]	[1]	[3]	[L]
Situation 8 Homo	4	9	1	10	45	12	4/15	715	37	220	0.30769231	3	1	N
	[4]	[9]	[1]	[10]	[45]	[12]	[4/15]	[10,000]	[37]	[3,439]	[0.3439]	[3]	[1]	[N]
Brown-Kruse and Hummels (1993)														
High Multiplier	4	1	1	2	1	0.5	0.5	5	5	2	0.4	1	3	L
	[4]	[1]	[1]	[2]	[1]	[0.5]	[0.5]	[16]	[5]	[5]	[0.3125]	[1]	[3]	[L]
Low Multiplier	4	1	1	2	1	0.3	0.3	5	5	4	0.8	3	1	Y
	[4]	[1]	[1]	[2]	[1]	[0.3]	[0.3]	[16]	[5]	[15]	[0.9375]	[3]	[1]	[Y]
Buckley and Croson (2006)	4	25;50	1	26;51	1	0.5	0.5	465,426	151	76	0.00016329	1	3	L
Burlando and Guala (2005)														
Strategy	4	200	1	201	1	0.5	0.5	70,058,751	801	201	0.00000287	1	3	-
Repeated	4	20	1	21	1	0.5	0.5	10,626	81	21	0.00197628	1	3	H
Cherry et al. (2005)														
Homo 10	4	10	1	11	1	0.5	0.5	1,001	41	11	0.01098901	1	3	L
Homo 20	4	20	1	21	1	0.5	0.5	10,626	81	21	0.00197629	1	3	L
Homo 30	4	30	1	31	1	0.5	0.5	46,376	121	31	0.00066845	1	3	L
Homo 40	4	40	1	41	1	0.5	0.5	135,751	161	41	0.00030202	1	3	L
Hetero 10; 20; 30; 40	4	10-40	1	11-41	1	0.5	0.5	293,601	101	101	0.00034400	1	3	L
Cinyabuguma et al. (2005)														
Base	16	10	1	11	1	0.2	0.2	5,311,735	161	1,001	0.00018845	4	12	L
Expul. Green	≤ 16	10	1	11	1	0.2	0.2	$\leq 5,311,735$	≤ 161	$\leq 1,001$	≥ 0.000188	4	≤ 12	Y
Expul. Blue	0 - 5	5	1	6	1	0.2	0.2	-	-	-	-	-	-	N
Expul. Blue	≥ 6	5	1	6	1	0.2	0.2	≥ 462	≥ 31	≥ 126	≤ 0.272727	4	≥ 2	N

Table 3: Continued

Author(s)	n	B	ϵ	m	α	β	$MPCR$	NOA	$NOWL$	$NOPA$	Pareto-ratio	n_{max}	n_{min}	PO
Denant-Boemont et al. (2007)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	H
Fehr and Gächter (2000)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	Y
	[4]	[20]	[1]	[21]	[1]	[0.4]	[0.4]	[194,481]	[81]	[2,481]	[0.012757]	[2]	[2]	[Y]
Fischbacher et al. (2001)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	-
Fischbacher and Gächter (2010)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	L
Gächter and Thöni (2005)	3	20	1	21	1	0.6	0.6	1,771	61	21	0.01185771	1	2	Y
	[3]	[20]	[1]	[21]	[1]	[0.6]	[0.6]	[9,261]	[61]	[61]	[0.0065868]	[1]	[2]	[Y]
Harbaugh and Krause (2000)														
Low-MPCR	6	5	1	6	1	1/3	1/3	462	31	21	0.04545455	2	4	L
High-MPCR	6	5	1	6	1	2/3	2/3	462	31	6	0.01298701	1	5	L
Herrmann and Thöni (2009)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	-
Houser and Kurzban (2002)	4	50	1	51	1	0.5	0.5	316,251	201	51	0.00016126	1	3	L
Isaac and Walker (1988)														
4L	4	62	1	63	0.01	0.003	0.3	720,720	249	43,680	0.06060606	3	1	H
4H	4	25	1	26	0.01	0.0075	0.75	23,751	101	26	0.00109469	1	3	L
10L	10	25	1	26	0.01	0.003	0.3	183,579,396	251	3,276	0.00001785	3	7	L
10H	10	10	1	11	0.01	0.0075	0.75	184,756	101	11	0.00005954	1	9	L
Isaac et al. (1994)														
4L	4	50	1	51	0.01	0.003	0.3	316,251	201	23,426	0.07407407	3	1	H
4H	4	50	1	51	0.01	0.0075	0.75	316,251	201	51	0.00016126	1	3	L
10L	10	50	1	51	0.01	0.003	0.3	75,394,027,566	501	23,426	0.00000031	3	7	L
10H	10	50	1	51	0.01	0.0075	0.75	75,394,027,566	501	51	$6.76 \cdot 10^{-10}$	1	9	L
40LL	40	50	1	51	0.01	0.0003	0.03	$5.9871 \cdot 10^{25}$	2,001	$1.49 \cdot 10^{23}$	0.00249533	33	7	H
40L	40	50	1	51	0.01	0.003	0.3	$5.9871 \cdot 10^{25}$	2,001	23,426	$3.91 \cdot 10^{-22}$	3	37	L
40H	40	50	1	51	0.01	0.0075	0.75	$5.9871 \cdot 10^{25}$	2,001	51	$8.52 \cdot 10^{-25}$	1	39	L
100L	100	50	1	51	0.01	0.003	0.3	$2.0129 \cdot 10^{40}$	5,001	23,426	$1.17 \cdot 10^{-36}$	3	97	L
100H	100	50	1	51	0.01	0.0075	0.75	$2.0129 \cdot 10^{40}$	5,001	51	$2.53 \cdot 10^{-39}$	1	99	L
Keser and van Winden (2000)	4	10	1	11	10	5	0.5	1,001	41	11	0.01098901	1	3	L
	[4]	[10]	[1]	[11]	[10]	[5]	[0.5]	[14,641]	[41]	[41]	[0.0028004]	[1]	[3]	[L]
Kurzban and Houser (2001)	4	50	1	51	0.01	0.005	0.5	316,251	201	51	0.00016126	1	3	L

Table 3: Continued

Author(s)	n	B	ϵ	m	α	β	$MPCR$	NOA	$NOWL$	$NOPA$	Pareto-ratio	n_{max}	n_{min}	PO
Leuthold (1993)	73	10,000	1^{**}	10,001	0.05	1/730	2/73	$2.9289 \cdot 10^{186}$	730,001	$2.87 \cdot 10^{102}$	$9.81 \cdot 10^{-85}$	36	37	-
Masclet et al. (2003)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	Y
Masclet et al. (2009)														
Sim. Low	4	10	1	11	1	0.5	0.5	1,001	41	11	0.01098901	1	3	L
Sim. High	8	10	1	11	1	0.5	0.5	43,758	81	11	0.00025138	1	7	L
McCorkle and Watts (1996)	49	100	100	2	0.05	1/490	2/49	50	50	25	0.5	24	25	-
Messer et al. (2007)	7	100	1^{**}	101	1	15/70	0.21	26,075,972,546	701	4,598,126	0.00017634	4	3	Y
Nikiforakis (2008)	4	20	1	21	1	0.4	0.4	10,626	81	231	0.02173913	2	2	H
Pickhardt (2005)														
Session I (block 1+2)	6	2	1	3	4	1	0.25	28	13	10	0.35714286	3	3	Y
Session L (block 1+2)	13	2	1	3	4	1	0.25	105	27	10	0.09523810	3	10	Y
Rege and Telle (2004)	10	150	1	151	1	0.2	0.2	$2.2741 \cdot 10^{15}$	1,501	22,533,126	$9.91 \cdot 10^{-9}$	4	6	Y
Tan (2008)														
Homo	4	10	1	11	1	0.6	0.6	1,001	41	11	0.01098901	1	3	Y
	[4]	[10]	[1]	[11]	[1]	[0.6]	[0.6]	[14,641]	[41]	[41]	[0.0028004]	[1]	[3]	[Y]
Hetero	4	10	1	11	1	0.3	0.3;0.9	4,356	281	76	0.01744720	-	-	N
	[4]	[10]	[1]	[11]	[1]	[0.3]	[0.3;0.9]	[14,641]	[281]	[141]	[0.0096305]	[-]	[-]	[N]
Weimann (1994)	5	100	1^{*}	101	0.5	0.25	0.5	96,560,646	501	101	0.00001046	1	4	L
Wilson and Sell (1997)	6	40	1	41	0.01	0.003	0.3	9,366,819	241	12,341	0.00131752	3	3	L

Note: *All figures refer to the first scenario, except those given in square brackets which refer to the second scenario. n denotes the number of human subjects in a group, B denotes the budget endowment or income of a subject, ϵ denotes the smallest unit in which the budget may be spend (e.g. one token). Note that ** indicates that the original ϵ is one-token cent (0.01) in the cases of Leuthold (1993) and Messer et al. (2007), with original budgets of 100 tokens and 1 token, respectively. Likewise in the case of Weimann (1994) * denotes an original ϵ of 0.1, with an original budget of 10 tokens. These amendments ensure that both ϵ and $B \in \mathbb{N}$ in case where these parameters are originally measured on different scales. Moreover, m denotes the number of alternatives which a subject may have in a round, α and β denote parameters of the payoff function, MPCR denotes marginal per capita return of a contribution to the public good, NOA denotes number of feasible allocations, NOWL denotes the number of unique welfare levels, NOPA denotes the number of Pareto-optimal allocations, Pareto-ratio denotes the ratio NOPA/NOA, n_{max} denotes the maximal number of free-riders tolerated by the Pareto-optimality concept, where free-riders are defined as those who do not fully contribute their budget to the public good, n_{min} denotes the minimum number of full contributors required by the Pareto-optimality concept,and PO denotes Pareto-optimal allocations, where '-' denotes a lack of data, L denotes a low potential for Pareto-optimal allocations, H denotes a high potential for Pareto-optimal allocations, N denotes that there are no Pareto-optimal allocations and Y denotes that there are one or more Pareto-optimal allocations.*

To begin with, we consider Brown-Kruse and Hummels (1993) who apply a preplay communication and interaction mechanism (cheap talk) in combination with pure male and female groups. In their low multiplier (i.e., low $MPCR$) treatment only one full contributor is required for a Pareto-optimal allocation (see Table 3). Moreover, since they consider a binary decision space, any contributor is a full contributor. Figure 4 (1993, 263) indicates that all males and almost all females have at least contributed in one round, many in more than one round and about 25 percent of the males in all rounds. In addition, Figure 2 and Table 3 (1993, 260-262) reveal that the mean contribution rate for males (females) was 93.8 (75) percent in the first round, which indicates 15 (12) full contributors in the four male (female) groups, respectively. Hence, the first rounds of the four male groups and at least three of the four first rounds of the female groups must have been Pareto-optimal. In fact, we cannot rule out that all rounds of the four male groups may have been Pareto-optimal.

Pickhardt (2005) considers just a simple face-to-face communication mechanism (cheap talk) between blocks of five rounds. In several sessions the cheap talk mechanism leads to a full contribution environment that is maintained over several rounds and, therefore, to Pareto-optimality. Moreover, it follows from the results presented for session I, round six (2005, 156, Fig. 3), which is the first round after the communication break, that the average contribution was about 1.67 tokens, which indicates that in this round 10 tokens were put into the public account. Given the parameter values presented in Table 3, we can infer that either five subjects have provided their entire endowment of two tokens to the public account or that four subjects have provided their entire endowment while the two remaining subjects provided one token each. Hence, in both cases $n_0 \geq n_{min}$ was fulfilled with either $5 > 3$ or $4 > 3$ and the allocation in round six was Pareto-optimal. Likewise, for session L it follows that the average contribution in round six was about 1.76 tokens, which indicates 23 tokens in the public account. Thus, there must have been at least ten full contributors and no more than three free-riders. Yet, during the following four rounds the average contribution path in both sessions shows the typical decline pattern. But would subjects be prepared to maintain such an allocation over several periods if they were aware of its Pareto-optimally?

Next, we consider results presented by Messer et al. (2007, 1790). They consider eight treatments that represent alternative combinations of three contextual factors: cheap talk, voting, and the status quo of the donation. According to Table 3, in their case an allocation would be Pareto-optimal if three or more subjects contribute their entire budget. Hence, the lowest average contribution that may be compatible with a Pareto-optimal allocation amounts to an average contribution level of $(3/7) \approx 0.429$. Inspection of the results presented for the eight treatments shows that the average contribution level in the first round is in one case (Treatment 5) in the range of 43 percent, but substantially higher in the remaining seven treatments. Thus, each initial allocation in each session of the eight treatments may have been Pareto-optimal. Moreover, they report that in treatment eight the simultaneous introduction of all three contextual factors results in voluntary contributions remaining at 100 percent over ten rounds in four of five sessions, which clearly indicates that these four sessions were Pareto-optimal. However, with respect to the remaining treatments it is not possible to infer whether or not a round of a certain session was actually Pareto-optimal, because results for each treatment show the average over five sessions and each subject has 101 alternatives to choose from. In particular, it is worth noting that even if all subjects would contribute 99 cent of their one dollar budget in a particular round, this round would not be Pareto-optimal because of $n_0 = 0$ and, thus, $n_0 \geq n_{min}$ would not hold. This notwithstanding, we can conclude that if the average contribution level is below $(3/35) \approx 0.085$, not one of the five sessions could have been Pareto-optimal. Yet, this is not the case in any of the ten rounds in any of the eight treatments.

Rege and Telle (2004) apply a social approval mechanism in combination with a framing mechanism in an one-shot game. In the approval treatment decisions are made anonymously, but once all subjects have made their decision each subject has to present his contribution to the public account in front of all other group members. In the no-approval treatment the procedure assures full anonymity. The framing mechanism consists of a language that suggests associations to social and internalized norms for cooperation (associative treatment) versus neutral language (non-associative treatment). For each of the four possible treatment combinations two sessions with ten subjects are conducted. Results (2004, 1634, Fig. 3) show that in the associative / approval treatment treatment there are 14 full contributors, three partial contributors and three non-contributors (full free-riders). Table 3 indicates that six full contributors are required in this game. Hence, at least one of these sessions must have been Pareto-optimal because there is no feasible allocation of the set of full contributors that assigns less than six full contributors to both sessions. Also, it cannot be ruled out that both sessions were Pareto-optimal. The same analysis holds true for the non-associative / approval treatment were 12 subjects fully contribute to the public account. However, with respect to the non-associative / no-approval treatment none of the two sessions could have been Pareto-optimal and regarding the associative / no-approval treatment no definite conclusion can be drawn with respect to Pareto-optimality.

Fehr and Gächter (2000) apply a punishment mechanism in combination with a partners versus strangers treatment. Table 3 indicates that in their case only two full contributors (50 percent) are required for Pareto-optimality. Results (2000, Fig. 1 to 4) show that average contributions in the punishment treatment are always above 50 percent of the endowment, except in the some rounds of the stranger treatment of session three. Moreover, the number of full contributors in the final round is in the range of 20 percent in the strangers / punishment treatment, but 80 percent in the partner / punishment treatment. Hence, of the 40 subjects in the partner / punishment treatment, 32 were full contributors in the final round. This indicates that the allocation in all ten groups could have been Pareto-optimal, but at least in eight groups the allocation in the final round must have been Pareto-optimal. The punishment treatment of Fehr and Gächter (2000) has been replicated, for example by Masclet et al. (2003), Denant-Boemont et al. (2007), and Nikiforakis (2008), who all confirm the findings of Fehr and Gächter (2000). However, Denant-Boemont et al. (2007) and Nikiforakis (2008) also find that counter punishment decreases contributions and, therefore, reduces the potential for achieving Pareto-optimal allocations.

Burlando and Guala (2005) employ the procedure of Fischbacher et al. (2001) by first running an one-shot linear public goods game with a view to classify subjects into behavioral types. They identify four types: free riders, reciprocators, cooperators and noisy. Next they run a repeated linear public goods game with two different treatments. In their heterogeneous treatment they randomly allocate subjects to groups of four, whereas in their homogeneous treatment they match subjects types into homogeneous groups of four. Hence, these authors essential test effects of a Tiebout mechanism in a linear public goods game. According to Table 3 a Pareto-optimal allocation would require three full contributors and, thus, an average contribution level of at least 75 percent or 15 tokens. It follows from their results (2005, 47, Fig. 4) that free rider groups are below the average of 15 tokens in all 20 rounds, whereas cooperators groups are in 14 rounds above that average and reciprocators groups are always substantially above that average, except in the final round. But again, since results show the average over several groups and each subject has 21 alternatives to choose from, it is not possible to infer whether or not a single round was actually Pareto-optimal.

Gächter and Thöni (2005) run a similar design with a few differences in parameters and some other aspects. Their results for the homogeneous treatments are even stronger than those of Burlando and Guala (2005) and, in some cases, definitely include Pareto-optimal allocations, because in their P-treatment several rounds display a mean contribution level of 100 percent. Hence, in both cases, results clearly show that the Tiebout mechanism increases the potential for achieving Pareto-optimal allocations in linear public goods games.

Cinyabuguma et al. (2005) show a similar result, but they use instead a Buchanan club mechanism. They run a baseline treatment and an expulsion treatment. In the expulsion treatment subjects are initially in a green group and can see the contributions made in the current and the two previous periods of all other subjects. They can also cast a vote for removing a certain subject from the green group. If half or more of the current members of the green group vote to expel a group member, that subject is moved to the blue group for the remaining rounds. As the budget for blue group members is much lower than for green group members (see Table 3), being removed from the green group implies a penalty. According to Table 3, in the initial round a Pareto-optimal allocation would require 12 full contributors and, thus, an average contribution level of at least 75 percent or 7.5 tokens. Results (2005, 1426, Fig. 1) show that the average contribution level of the green group is always substantially above that level, whereas in the baseline treatment it is always below the average of 7.5 tokens. Moreover, in their treatment EE2 the average contribution rate over the two sessions is 94.8 percent, which indicates that 303 tokens where contributed. Yet, if we assume that all 32 subjects contributed nine of their ten tokens, we just get a total of 288 tokens. Thus, there must have been at least 15 full contributors and the number of full contributors would increase for every subject who has contributed less than nine tokens. This indicates a very high potential for at least one Pareto-optimal allocation in the first round of the EE2 treatment. This potential increases even further as some subjects are expelled in the following rounds and because the average contribution rate continues to increase as the game proceeds. From round ten trough thirteen average contribution reaches 100 percent, so that these rounds must have been Pareto-optimal. Hence, the results show that the Buchanan club mechanism allows for achieving Pareto-optimal allocations in linear public goods games. In fact, the Buchanan club mechanism may reinforce an achieving of Pareto-optimal allocations. First, since expulsion of group members reduces n and n_{max} is given by α, β, γ and δ, expulsion reduces the number of full contributors n_{min} needed for a Pareto-optimal allocation. Second, if the lowest contributors of a group are expelled (as observed by the authors, 2005) and expulsion implies a penalty, subjects in the green group may even be pushed toward full contribution (as in their EE2 treatment), which again helps to establish Pareto-optimality.

Finally, Bazart and Pickhardt (2011) apply a carrot and stick mechanism to establish a full contribution environment in a linear public goods game that is embedded in a tax evasion experiment. If subjects are audited, a penalty is due for those who do not fully contribute. In addition, in the second and third block of the experiment, those who do fully contribute may participate in a lottery scheme, in which one subject can win a substantial private payoff. According to Table 3, in their case a Pareto-optimal allocation requires four full contributors. Hence, if there are less than four full contributors per block of six rounds, none of the six rounds could have been Pareto-optimal. In fact, this can be verified as the results they present show, among other things, the number of full contributors per block of six rounds (2011, Table 2). In the first block, there are less than four full contributors in four of the nine sessions, but this is never the case in any of the nine session in the second or third block. Further, with 24 full contributors all six rounds could have been Pareto-optimal, but at least three rounds must have been Pareto-optimal. The number of 24 full contributors was reached or exceeded in two sessions of the second block and in five sessions of the third block, but it was never reached in the first block. Therefore, the carrot and stick mechanism not only leads to a significant increase in the average contribution level, but also to a significant increase in the number of Pareto-optimal allocations.

To summarize, Pareto-optimal allocations are traceable in a wider set of linear public goods games. Yet, it seems that a contributions enhancing mechanism is required to achieve and maintain Pareto-optimal allocations. In this context an interesting research question is whether and to what extent information about the actual Pareto-optimality of an allocation matters for maintaining this allocation. Another research question of interest is whether preplay information about the Pareto-optimality of certain allocations enhances initial contribution rates so that Pareto-optimal outcomes are achieved already in the first round.

5.3 The Decay of Voluntary Contributions

We now briefly demonstrate how the frequently observed decay of voluntary contributions in linear public goods games with human subjects can be analyzed with the calculation procedure and tables introduced earlier on. Typically, the average voluntary contribution path often starts in the range of 40 to 60 percent of total endowment and then drops down to much lower levels with repetitions, although it rarely drops down to zero voluntary contributions (e.g. see Ledyard 1995; Zelmer 2003; Cox and Sadiraj 2007). Researchers have provided a number of alternative explanations for this behavior pattern. For example, voluntary contributions may be due to *noise* (e.g. errors, subject confusion, etc.), so that the decay of voluntary contributions may be due to learning effects (e.g. see Houser and Kurzban 2002). An alternative explanation is that subjects have *other-regarding preferences* that are either independent from the behavior of others (e.g. altruism, warm-glow, etc.), or that do depend on the behavior of others (e.g. imperfect conditional cooperation, cooperative gain seeking, etc.; see Brandts and Schram 2001; Figuières et al. 2009; Fischbacher and Gächter 2010; Pickhardt 2010, among others).

For simplicity, we now assume that the game represented by Table 1 prevails. It follows from inspection of Table 1 that alternative *NC* does not involve any risk taking and guarantees a save payoff of at least eight tokens. Therefore, the payoff of eight tokens may serve as a benchmark against any other alternative that involves some risk taking due to voluntary contributions, here alternatives *FC* and *PC*. Based on this benchmarking we can distinguish various subsets of the set of 21 allocations in Table 1. These subsets are: *the benchmark set*, which contains just allocation 1; *the non-profit set*, which includes allocations $2, 3, 4, 5, 6, 7, 8, 9, 15$, and 19, because contributing does not lead to a payoff higher than the benchmark; *the profit set*, which includes allocations $10, 11, 12, 13, 14, 16, 17, 18, 20$, and 21, because contributing generates for at least one subject a payoff in excess of the benchmark. The profit set can be further distinguished into two subsets *the impure profit set*, which includes allocations $11, 12, 13, 14, 16, 17$, and 18 and *the pure profit set*, which includes allocations $10, 20$, and 21. Note that the pure profit set is characterized by the fact that each subject receives a payoff that exceeds the benchmark of eight tokens.

By using such allocation sets experimenters may develop designs that test behavioral hypothesis in line with alternative explanations provided in the literature and mentioned above. In any case, it seems that the first round allocation and the set of behavioral types in the group under consideration are of paramount importance for the voluntary contributions path (or allocations path) that may emerge in a finitely repeated linear public goods game. Pickhardt (2010) and Hokamp (2011) analyze these aspects in an agent-based linear public goods model.

5.4 Heterogeneous Endowments and Contribution Behavior of the Rich and Poor

In this subsection we show how the calculation procedure can be applied to the analysis of the behavior of rich and poor subjects in linear public goods games with heterogeneous income settings. In addition, we show how Pickhardt's table (2003; 2005) changes when income heterogeneity is introduced.

In particular, we further analyze two papers with heterogeneous income settings, which are Cherry et al. (2005) and Buckley and Croson (2006). Table 3 indicates that Cherry et al. (2005) consider a case with four different income levels (hetero) and, in addition, each of these four levels in a homogeneous setting (homo) for matters of comparison. In contrast, Buckley and Croson (2006) report the results of a heterogeneous setting with two different income levels. Table 3 also indicates that Pickhardt's tables for Cherry et al. (2005) and Buckley and Croson (2006) are both too large to be visualized.

For simplicity, we therefore consider a case that mimics Buckley and Croson (2006) as closely as possible. We assume a group of four agents, of which two agents are endowed with a budget of just one token each, henceforth called poor agents (first scenario: $p = 1$, second scenario: $p = 1, 2$), and the remaining two agents have a budget of two tokens each, henceforth called rich agents (first scenario: $p = 2$, second scenario: $p = 3, 4$). Otherwise, we assume the same parameter values as in the original case, which leads to the payoff function $U(y_i, X) = y_i + 0.5X$ for all four agents, according to (2). Then, according to (61), with respect to the first scenario we get $\mathfrak{F} = \{(1; 0.5; 1; 1; 1; 2; 2), (1; 0.5; 1; 1; 2; 3; 2)\}$, and with respect to the second scenario we get $\mathfrak{F} = \{(1; 0.5; 1; 1; 1; 2; 1), \{(1; 0.5; 1; 1; 1; 2; 1), (1; 0.5; 1; 1; 2; 3; 1), (1; 0.5; 1; 1; 2; 3; 1)\}$. It is worth mentioning that although different total budgets occur, the Gini coefficient g is identical in both cases, with $g \approx 0.167$. Table 4 is then obtained by generating tables for each subgroup of agents endowed with the same budget separately and by matching these tables into a joint table. Thus, Table 4 visualizes, for both scenarios, the set of feasible allocations $\mathfrak{D}$ and the set of feasible levels of welfare $\mathfrak{W}$ for the adjusted Buckley and Croson (2006) case.

Applying (66) yields $NOA = 18$ [$NOA = 36$], applying (67) yields $NOWL = 7$ [$NOWL = 7$] and according to (68) we get $NOA(0) = 1$ [$NOA(0) = 1$] and $NOA(1) = 3$ [$NOA(1) = 6$] and, thus, $NOPA = 4$ [$NOPA = 7$], where square brackets denote results for the second scenario. Inspection of Table 4 confirms these results.

To analyze the incentive structure for contributions to the public good, we introduce for each subgroup of agents endowed with the same tuple $(\alpha^p; \beta^p; \gamma^p; \delta^p; B^p; m^p)$ a loss-ratio (LR) and a profit-ratio (PR). The latter are defined as the number of allocations where at least one agent of the relevant subgroup contributes to the public good and these contributing agents get a strictly lower (higher) payoff than in Nash equilibrium (i.e. allocation 1) $NOA(lo)$ ($NOA(hi)$), respectively, divided by the number of allocations where at least one agent of the relevant subgroup contributes to the public good $NOA(con)$. Again, we use superscript $p \in \{1, \ldots, P\}$ to indicate different agent

Table 4: Set of Feasible Allocations in the Heterogeneous Budgets Case

Allocation	$(n_{0,RI}, n_{0,PO}) \times U_{i,0}$	$(n_{1,RI}, n_{1,PO}) \times U_{i,1}$	$n_{2,RI} \times U_{i,2}$	X	Welfare	CA
1	—	$(-, 2) \times 1.0$	2×2.0	0	6	0
2	—	$(1, 2) \times 1.5$	1×2.5	1	7	1
3	$(-, 1) \times 0.5$	$(-, 1) \times 1.5$	2×2.5	1	7	1
4	—	$(2, 2) \times 2.0$	—	2	8	0
5	$(1, -) \times 1.0$	$(-, 2) \times 2.0$	1×3.0	2	8	1
6	$(-, 1) \times 1.0$	$(1, 1) \times 2.0$	1×3.0	2	8	3
7	$(-, 2) \times 1.0$	—	2×3.0	2	8	0
8	$(1, -) \times 1.5$	$(1, 2) \times 2.5$	—	3	9	1
9	$(-, 1) \times 1.5$	$(2, 1) \times 2.5$	—	3	9	1
10	$(1, 1) \times 1.5$	$(-, 1) \times 2.5$	1×3.5	3	9	3
11	$(-, 2) \times 1.5$	$(1, -) \times 2.5$	1×3.5	3	9	1
12	$(2, -) \times 2.0$	$(-, 2) \times 3.0$	—	4	10	0
13	$(1, 1) \times 2.0$	$(1, 1) \times 3.0$	—	4	10	3
14	$(-, 2) \times 2.0$	$(2, -) \times 3.0$	—	4	10	0
15	$\mathbf{(1, 2) \times 2.0}$	—	$\mathbf{1 \times 4.0}$	**4**	**10**	**1**
16	$\mathbf{(2, 1) \times 2.5}$	$\mathbf{(-, 1) \times 3.5}$	—	**5**	**11**	**1**
17	$\mathbf{(1, 2) \times 2.5}$	$\mathbf{(1, -) \times 3.5}$	—	**5**	**11**	**1**
18	$\mathbf{(2, 2) \times 3.0}$	—	—	**6**	**12**	**0**

Note: *Allocation denotes the number of allocation, $n_{0,RI}$ ($n_{1,RI}$, $n_{2,RI}$) denotes the number of rich agents who choose PC_0 (PC_1, PC_2), respectively, $n_{0,PO}$ ($n_{1,PO}$) denotes the number of poor agents who choose PC_0 (PC_1), respectively, where constraints $n_{RI} = 2 = n_{0,RI} + n_{1,RI} + n_{2,RI}$, $n_{PO} = 2 = n_{0,PO} + n_{1,PO}$ and $n = 4 = n_{RI} + n_{PO}$ are fulfilled and $U_{i,0}$, ($U_{i,1}$, $U_{i,2}$) denotes the individual payoff, which any agent who has selected PC_0 (PC_1, PC_2) will receive, respectively, $(n_{0,RI}, n_{0,PO}) \times U_{i,0}$ ($(n_{1,RI}, n_{1,PO}) \times U_{i,1}$, $n_{2,RI} \times U_{i,2}$) denotes total payoff of the group of these agents, respectively (results not displayed), X denotes the total quantity of the public good, Welfare denotes overall payoff of the whole group, CA denotes the number of clone allocations, and allocations in bold denote Pareto-optimal allocations.*

subgroups. Thus, we formally obtain,

$$\mathrm{LR}^p = \frac{NOA(lo)^p}{NOA(con)^p}, \quad \forall p \in \{1, \ldots, P\} \tag{79}$$

and

$$\mathrm{PR}^p = \frac{NOA(hi)^p}{NOA(con)^p}, \quad \forall p \in \{1, \ldots, P\} \tag{80}$$

In Table 4 and Table 5 we identify allocations 3, 6, 7, 9, 10, 11, 13, 14, 15, 16, 17 and 18, as allocations where at least one poor agent contributes to the public good, and, thus, we obtain $NOA(con)^1 = 12$ [$NOA(con)^1 = NOA(con)^2 = 18$] (see Table 4 and Table 5, column $NOA(con)_{PO}$). Yet, there is only one allocation, which is allocation 3, where contributing leads to a lower payoff for poor agents, which yields $NOA(lo)^1 = 1$ [$NOA(lo)^1 = NOA(lo)^2 = 1$] (see column $NOA(lo)_{PO}$). In contrast, allocations 9, 10, 11, 13, 14, 15, 16, 17 and 18 have the property that contributing leads to a higher payoff for poor agents than in Nash equilibrium, which yields $NOA(hi)^1 = 9$ [$NOA(hi)^1 = NOA(hi)^2 = 14$] (see column $NOA(hi)_{PO}$). Therefore, for poor agents we obtain $\mathrm{LR}^1 = 1/12 \approx 0.083$ [$\mathrm{LR}^1 = \mathrm{LR}^2 = 1/18 \approx 0.056$] according to (79) and $\mathrm{PR}^1 = 9/12 = 0.75$ [$\mathrm{PR}^1 = \mathrm{PR}^2 = 14/18 \approx 0.778$] according to (80). With respect to rich agents, we get $NOA(con)^2 = 15$ [$NOA(con)^3 = NOA(con)^4 = 24$], which includes allocations 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18 (see column $NOA(con)_{RI}$), $NOA(lo)^2 = 4$ [$NOA(lo)^3 = NOA(lo)^4 = 5$], which includes allocations 2, 5, 8 and 10 (see column $NOA(lo)_{RI}$), and $NOA(hi)^2 = 6$ [$NOA(hi)^3 = NOA(hi)^4 = 12$], which includes allocations 9, 11, 14, 16, 17 and 18, [8, 9, 11, 13, 14, 16, 17 and 18] (see column

$NOA(hi)_{RI}$). Hence, for rich agents we obtain $\text{LR}^2 = 4/15 \approx 0.267$ [$\text{LR}^3 = \text{LR}^4 = 5/24 \approx 0.208$] according to (79) and $\text{PR}^2 = 6/15 = 0.4$ [$\text{PR}^3 = \text{PR}^4 = 12/24 = 0.5$] according to (80). Table 5 visualizes and summarizes these results.

Table 5: Loss- and Profit-ratios in the Heterogeneous Budgets Case

Allocation	# Perm.	$NOA(con)_{PO}$	$NOA(lo)_{PO}$	$NOA(hi)_{PO}$	$NOA(con)_{RI}$	$NOA(lo)_{RI}$	$NOA(hi)_{RI}$
1	1 [1]	—[—]	— [—]	— [—]	— [—]	— [—]	— [—]
2	1 [2]	—[—]	— [—]	— [—]	1 [1]	1 [1]	— [—]
3	1 [2]	1 [1]	1 [1]	— [—]	— [—]	— [—]	— [—]
4	1 [1]	—[—]	— [—]	— [—]	1 [1]	— [—]	— [—]
5	1 [2]	—[—]	— [—]	— [—]	1 [1]	1 [1]	— [—]
6	1 [4]	1 [2]	— [—]	— [—]	1 [2]	— [—]	— [—]
7	1 [1]	1 [1]	— [—]	— [—]	— [—]	— [—]	— [—]
8	1 [2]	—[—]	— [—]	— [—]	1 [2]	1 [1]	— [1]
9	1 [2]	1 [1]	— [—]	1 [1]	1 [2]	— [—]	1 [2]
10	1 [4]	1 [2]	— [—]	1 [2]	1 [2]	1 [2]	— [—]
11	1 [2]	1 [2]	— [—]	1 [2]	1 [1]	— [—]	1 [1]
12	1 [1]	—[—]	— [—]	— [—]	1 [1]	— [—]	— [—]
13	1 [4]	1 [2]	— [—]	1 [2]	1 [4]	— [—]	— [2]
14	1 [1]	1 [1]	— [—]	1 [1]	1 [1]	— [—]	1 [1]
15	**1 [2]**	**1 [2]**	— [—]	**1 [2]**	**1 [1]**	— [—]	— [—]
16	**1 [2]**	**1 [1]**	— [—]	**1 [1]**	**1 [2]**	— [—]	**1 [2]**
17	**1 [2]**	**1 [2]**	— [—]	**1 [2]**	**1 [2]**	— [—]	**1 [2]**
18	**1 [1]**	**1 [1]**	— [—]	**1 [1]**	**1 [1]**	— [—]	**1 [1]**
Σ	18 [36]	12 [18]	1 [1]	9 [14]	15 [24]	4 [5]	6 [12]
Loss- and Profit-ratio		Poor	$\frac{1}{12}$ [$\frac{1}{18}$]	$\frac{9}{12}$ [$\frac{14}{18}$]	Rich	$\frac{4}{15}$ [$\frac{5}{24}$]	$\frac{6}{15}$ [$\frac{12}{24}$]

Note: *All figures refer to the first scenario, except those given in square brackets which refer to the second scenario. Allocation denotes the number of allocation corresponding to Table 4, # Perm. denotes the number of feasible permutations of allocations, $NOA(con)_{PO}$ ($NOA(lo)_{PO}$, $NOA(hi)_{PO}$, $NOA(con)_{RI}$, $NOA(lo)_{RI}$, and $NOA(hi)_{RI}$) denotes the number of allocations where at least one agent of the relevant subgroup (poor agents (PO), rich agents (RI)) contributes to the public good (where at least one agent of the relevant subgroup (PO, RI) contributes to the public good and these contributing agents get a strictly lower (higher) payoff than in Nash equilibrium (i.e. allocation 1)), respectively, and allocations in bold denote Pareto-optimal allocations. The Σ-line indicates the column sum of $NOA(con)_{PO}$, $NOA(lo)_{PO}$, $NOA(hi)_{PO}$, $NOA(con)_{RI}$, $NOA(lo)_{RI}$, and $NOA(hi)_{RI}$. The last line from above presents Loss- and Profit-ratios for poor and rich agents.*

Now suppose that subjects contribute because they have other-regarding preferences, for example, that they are motivated by cooperative gain seeking. In this case a low loss-ratio combined with a high profit-ratio would indicate that cooperative gain seeking is associated with a comparatively low risk. Put differently, it seems reasonable to argue that in general a low loss-ratio combined with a high profit-ratio indicates a strong incentive to contribute to the public good, because there are only a few allocations where subjects may get a lower payoff than in Nash equilibrium, but many allocations where subjects may get a higher payoff. Thus, it follows from the calculated loss- and profit-ratios for the case of Table 4 that the poor (rich) agents have a stronger (weaker) incentive to contribute to the public good.

Table 6 summarizes the results. Inspection of Table 6, with respect to column R_A, which shows the actual voluntary contribution results in terms of percent of allocated income obtained by Buckley and Croson (2006), lines 1, 2, 4, and 5, and by Cherry et al. (2005), lines 13 to 24, makes it clear that these results comply with the incentive structure that follows from of the loss- and profit-ratios. In particular, the poor (rich) subjects of the Buckley and Croson setting contribute a higher (lower) percentage share of their allocated income endowment and the same is true for the Cherry et al. setting (lines 21 to 24), which compares to the Buckley and Croson case. Interestingly, the loss-

Table 6: Contribution Behaviour of the Rich and Poor

Line	$\mathfrak{G}$	P	TB	g	α	β	$MPCR$	NOA	$NOWL$	$NOPA$	Pareto-ratio	n_{max}	n_{min}	LR	PR	R_A	R_W (R_E)
Buckley and Croson (2006), First Scenario																	
1	{(25; 26; 2),	2	150	0.1667	1	0.5	0.5	465,426	151	76	0.00016329	1	3	0.024	0.973	0.285	0.328
2	(50; 51; 2)}													0.289	0.691	0.173	0.153
3														*0.201*	*0.785*		
Buckley and Croson (2006), Second Scenario																	
4	{(25; 26; 2),	4	150	0.1667	1	0.5	0.5	1,758,276	151	151	0.00008588	1	3	0.012	0.986	0.285	0.328
5	(50; 51; 2)}													0.149	0.840	0.173	0.153
6														*0.1037*	*0.889*		
Adjusted Example based on Buckley and Croson (2006), First Scenario																	
7	{(1; 2; 2),	2	6	0.1667	1	0.5	0.5	18	7	4	0.22222222	1	3	0.083	0.75		
8	(2; 3; 2)}													0.267	0.4		
9														*0.206*	*0.517*		
Adjusted Example based on Buckley and Croson (2006), Second Scenario																	
10	{(1; 2; 2),	4	6	0.1667	1	0.5	0.5	36	7	7	0.19444444	1	3	0.056	0.778		
11	(2; 3; 2)}													0.208	0.5		
12														*0.157*	*0.593*		
Cherry et al. (2005), Homo, First Scenario																	
13	{(10; 11; 4)}	1	40	0	1	0.5	0.5	1,001	41	11	0.01098901	1	3	0.189	0.745	0.5	0.425
14	{(20; 21; 4)}	1	80	0	1	0.5	0.5	10,626	81	21	0.00197629	1	3	0.180	0.786	0.469	0.475
15	{(30; 31; 4)}	1	120	0	1	0.5	0.5	46,376	121	31	0.00066845	1	3	0.176	0.801	0.375	0.367
16	{(40; 41; 4)}	1	160	0	1	0.5	0.5	135,751	161	41	0.00030202	1	3	0.174	0.809	0.381	0.381
Cherry et al. (2005), Homo, Second Scenario																	
17	{(10; 11; 4)}	4	40	0	1	0.5	0.5	14,641	41	41	0.00280036	1	3	0.057	0.925	0.5	0.425
18	{(20; 21; 4)}	4	80	0	1	0.5	0.5	194,481	81	81	0.00041649	1	3	0.048	0.943	0.469	0.475
19	{(30; 31; 4)}	4	120	0	1	0.5	0.5	923,251	121	121	0.00013106	1	3	0.046	0.948	0.375	0.367
20	{(40; 41; 4)}	4	160	0	1	0.5	0.5	2,825,761	161	161	0.00005698	1	3	-	-	0.381	0.381
Cherry et al. (2005), Hetero, First and Second Scenario																	
21	{(10; 11; 1),	4	100	0.25	1	0.5	0.5	293,601	101	101	0.00034400	1	3	0.003	0.996	0.65	0.425
22	(20; 21; 1),													0.03	0.965	0.288	0.206
23	(30; 31; 1),													0.117	0.87	0.217	0.442
24	(40; 41; 1)}													0.279	0.701	0.188	0.234
25														*0.153*	*0.834*		

Note: *Line denotes the number of the line, $\mathfrak{G}$ denotes the last three relevant items of the tuple (B, m and f) that characterizes the differences between subgroups, P denotes the number of subgroups, TB denotes total budget, g denotes Gini coefficient, α and β denotes payoff parameters, MPCR is the marginal per capita return, NOA is the total number of allocations, NOWL is the number of welfare levels, NOPA is the number of Pareto-optimal allocations, Pareto-ratio denotes NOPA / NOA, n_{max} denotes the maximal number non-full-contributors (free-riders) that is compatible with the Pareto-optimality concept, n_{min} is the minimum number of full contributors required for a Pareto-optimal allocation, LR denotes loss-ratio, PR denotes profit-ratio, where figures in italics denote weighted loss-ratio and weighted profit-ratio, R_A denotes results based on allocated income, R_W denotes results based on wealth (Buckley and Croson 2006) and R_E denotes results based on earned income (Cherry et al. 2005).*

and profit-ratios and the results obtained by Buckley and Croson for a 25 vs. 50 tokens budgets case are practically identical with the Cherry et al. cases of 20 vs. 40 tokens budgets (see Table 6, line 1 and 2 vs. lines 22 and 24). Moreover, column R_W, Table 6, lines 1 and 2, shows that the same is true if the percentage of contributed wealth rather than income is considered in the Buckley and Croson case. Results of Cherry et al. with respect to earned rather than allocated income, column R_E, lines 21 to 24, also comply with the incentive structure of the loss- and profit-ratios, although the relation is much weaker. This notwithstanding, in general results seem to indicate that subjects do care for loss- and profit-ratios, which in turn suggests that contributing subjects may indeed be motivated by cooperative gain seeking.

Finally, it must be emphasized that the loss- and profit-ratios have been derived under the implicit assumption that each allocation in Table 4 occurs with the same probability regardless of the considered scenario. This assumption was made for simplicity because there exists no reasonable alternative and, thus, subjects in an experiment might intuitively do the same. It would be interesting, however, to run an experiment like the one on which Table 4 is based and to see with which frequency each allocation actually occurs. Given a sufficient number of runs, the loss- and profit-ratios could be recalculated on this basis and again compared to the contributed percentages of income by the rich and poor.

5.5 Heterogeneous Endowments and Neutrality

In this subsection we show how the calculation procedure can be used for analyzing redistribution effects in linear public goods games with heterogeneous income distributions. Redistribution effects have been studied theoretically with respect to non-linear public goods games by Warr (1983), Bergstrom et al. (1986) and experimentally by Maurice et al. (2009), among others. These authors find that a redistribution of income among contributors is neutral in the sense that it does not affect aggregate contributions to the public good.

However, with respect to linear public goods games it seems that redistribution effects in heterogeneous income settings have not been studied. Hence, in a first attempt, we make use of the loss- and profit-ratios introduced in the preceding subsection and, for matters of comparability, we also introduce weighted loss- and profit-ratios. That are,

$$\mathrm{LR} = \sum_{p=1}^{P} \frac{\mathrm{LR}^p \cdot f^p \cdot B^p}{TB} \tag{81}$$

and

$$\mathrm{PR} = \sum_{p=1}^{P} \frac{\mathrm{PR}^p \cdot f^p \cdot B^p}{TB}, \tag{82}$$

where TB denotes the total budget of the group of n agents, with $TB := \sum_{i=1}^{n} B_i$.

As in the previous subsection, the calculation is based on the simplifying assumption that allocations are uniformly distributed. Next, we consider the case of Cherry et al. (2005) again, but with some modifications. First of all, we fix the total budget to 100 tokens as in their original heterogeneous setting. We then consider a homogeneous setting where each of the four agents receives a budget of 25 tokens and regard the loss- and profit-ratios of this case as the benchmark for all weighted loss- and profit-ratios of redistribution cases that are based on a fixed total budget of 100 tokens. In particular, we analyze four redistribution cases and the original heterogeneous setting. Table 7 summarizes the results.

Inspection of Table 7 shows that the aggregate incentive structure for contributions to the public good, which is indicated by the weighted loss-and profit-ratios denoted in italics in Table 7, differs from the homogeneous case, 0.178 [0.047] and 0.795 [0.946], respectively, with respect to each redistribution case. Also, identical Gini coefficients do not necessarily imply identical aggregate incentive structures, which follows from a comparison of lines 12 [25] and 15 [28]. Therefore, in an experiment with human subjects, it seems reasonable to assume that actual contributions to the public good would differ in each redistribution case from those made in the homogeneous setting (Table 7, line 6 [19]). Moreover, since in all cases with heterogeneous income distributions there are higher weighted loss-ratios and lower weighted profit-ratios, one would expect that the contributions to the public good are lower in heterogeneous income cases compared to homogeneous income settings. Interestingly, Zelmer (2003) finds statistically evidence for this outcome. But, of course, this needs to be tested with an appropriate experimental design.

5.6 Heterogeneous MPCRs

To proceed, we illustrate the heterogeneous $MPCR$ setting with Tan (2008). In her experiment there are four agents, i.e. $n = 4$, of which two have a low $MPCR = 0.3$ and two have a high $MPCR = 0.9$. Also, her payoff function (see Tan 2008, 277, Eq. 2) differs from the one we use here (2). Further, each agent is endowed with a budget of ten tokens, which yields $B = 10$, $\forall i \in \mathfrak{I}_4$. The budget constraint (3) is identical for each agent, with $\gamma = \delta = 1$, $\forall i \in \mathfrak{I}_4$. The smallest possible unit in which the budget may be spend is one token, i.e. $\epsilon = 1$, and applying (4) yields $m = 11$, $\forall i \in \mathfrak{I}_4$. According to (5) we have $\zeta = \eta = 1$, $\forall i \in \mathfrak{I}_4$.

To get for each agent a payoff function according to (2), we modify Tan's original parameter setting *ceteris paribus* by assuming $\zeta = 1$ and $\eta = 3$ for agents with a high $MPCR$, which yields $\gamma = 1$ and $\delta = 1/3$, because for convenience we decided to keep $\epsilon = 1$ constant. This adaptation allows for applying the calculation procedures for NOA, $NOWL$ and $NOPA$ developed in section 4 to Tan's original paper. In the first scenario we have two subgroups, i.e. $P = 2$, which leads to $\mathfrak{F} := \{(1; 0.3; 1; 1; 10; 11; 2), (1; 0.3; 1; 1/3; 10; 11; 2)\}$. According to the second scenario we use $\mathfrak{F} := \{(1; 0.3; 1; 1; 10; 11; 1), (1; 0.3; 1; 1; 10; 11; 1), (1; 0.3; 1; 1/3; 10; 11; 1), (1; 0.3; 1; 1/3; 10; 11; 1)\}$, but again, these tables are to large to be visualized. However, results for both scenarios are displayed in Table 3.

For simplicity, we therefore provide an example that mimics the case of Tan (2008) as closely as possible. We assume that each agent is endowed with a budget of just two tokens, i.e. $B = 2$, which yields $m = 3$, $\forall i \in \mathfrak{I}_4$, and keep the remaining parameters unchanged. Therefore, we consider for the first scenario $\mathfrak{F} := \{(1; 0.3; 1; 1; 2; 3; 2), (1; 0.3; 1; 1/3; 2; 3; 2)\}$ and with respect to the second scenario we have $\mathfrak{F} := \{(1; 0.3; 1; 1; 2; 3; 1), (1; 0.3; 1; 1; 2; 3; 1), (1; 0.3; 1; 1/3; 2; 3; 1), (1; 0.3; 1; 1/3; 2; 3; 1)\}$. Table 8 visualizes, for both scenarios, the set of feasible allocations $\mathfrak{D}$ and the set of feasible levels of welfare $\mathfrak{W}$ for the adjusted Tan (2008) case. Regarding the first and second scenario, applying (69) yields $NOA = 36$ and $NOA = 81$, respectively. According to (64) both subgroups of agents fulfill the individualized necessary condition for a prisoner's dilemma situation, i.e. $0.3 < 1$ and $0.9 < 1$. In addition, for both subgroups the individualized sufficient condition holds for $2.0 < 4.8$ according to (65). Note that

Table 7: Redistribution Effects in Linear Public Goods Games

Line	$\mathfrak{G}$	P	TB	g	α	β	$MPCR$	NOA	$NOWL$	$NOPA$	Pareto-ratio	n_{max}	n_{min}	LR	PR	R_A	R_E
Cherry et al. (2005), Hetero, First and Second Scenario																	
1	{(10; 11; 1),	4	100	0.25	1	0.5	0.5	293,601	101	101	0.00034400	1	3	0.003	0.996	0.65	0.425
2	(20; 21; 1),													0.03	0.965	0.288	0.206
3	(30; 31; 1),													0.117	0.87	0.217	0.442
4	(40; 41; 1)}													0.279	0.701	0.188	0.234
5														*0.153*	*0.834*		
Redistribution Examples based on Cherry et al. (2005), First Scenario																	
6	{(25; 26; 4)}	1	100	0	1	0.5	0.5	23,751	101	26	0.00109469	1	3	0.178	0.795		
7	{(20; 21; 2),	2	100	0.1	1	0.5	0.5	114,576	101	51	0.00044512	1	3	0.042	0.949		
8	(30; 31; 2)}													0.186	0.791		
9														*0.129*	*0.854*		
10	{(10; 11; 1),	2	100	0.15	1	0.5	0.5	60,016	101	41	0.00068315	1	3	0.003	0.995		
11	(30; 31; 3)}													0.296	0.67		
12														*0.267*	*0.703*		
13	{(20; 21; 3),	2	100	0.15	1	0.5	0.5	72,611	101	61	0.00084009	1	3	0.067	0.92		
14	(40; 41; 1)}													0.277	0.702		
15														*0.151*	*0.833*		
16	{(10; 11; 2),	2	100	0.3	1	0.5	0.5	56,826	101	51	0.00089748	1	3	0.008	0.99		
17	(40; 41; 2)}													0.553	0.411		
18														*0.444*	*0.527*		
Redistribution Examples based on Cherry et al. (2005), Second Scenario																	
19	{(25; 26; 4)}	4	100	0	1	0.5	0.5	456,976	101	101	0.00022102	1	3	0.047	0.946		
20	{(20; 21; 2),	4	100	0.1	1	0.5	0.5	423,801	101	101	0.00023832	1	3	0.022	0.974		
21	(30; 31; 2)}													0.097	0.890		
22														*0.067*	*0.9237*		
23	{(10; 11; 1),	4	100	0.15	1	0.5	0.5	327,701	101	101	0.00030821	1	3	0.002	0.997		
24	(30; 31; 3)}													0.106	0.882		
25														*0.096*	*0.893*		
26	{(20; 21; 3),	4	100	0.15	1	0.5	0.5	379,701	101	101	0.00026600	1	3	0.025	0.971		
27	(40; 41; 1)}													0.2739	0.705		
28														*0.124*	*0.865*		
29	{(10; 11; 2),	4	100	0.3	1	0.5	0.5	203,401	101	101	0.00049656	1	3	0.004	0.995		
30	(40; 41; 2)}													0.290	0.692		
31														*0.233*	*0.752*		

Note: *Line denotes the number of the line,* $\mathfrak{G}$ *denotes the last three relevant items of the tuple (B, m and f) that characterizes the differences between the p-subgroups, P denotes the number of subgroups, TB denotes total budget, g denotes Gini coefficient,* α *and* β *denotes the payoff parameters, MPCR is the marginal per capita return, NOA is the total number of allocations, NOWL is the number of welfare levels, NOPA is the number of Pareto-optimal allocations, Pareto-ratio denotes NOPA / NOA,* n_{max} *denotes the maximal number non-full-contributors (free-riders) that is compatible with the Pareto-optimality concept,* n_{min} *is the minimum number of full contributors required for a Pareto-optimal allocation, LR denotes loss-ratio, PR denotes profit-ratio, where figures in italics denote weighted loss-ratio and weighted profit-ratio,* R_A *denotes results based on allocated income, and* R_E *denotes results based on earned income.*

Table 8: Set of Feasible Allocations in the Heterogeneous $MPCR$ Case

Allocation	$(n_{0,LOW}, n_{0,HIGH}) \times U_{i,0}$	$(n_{1,LOW}, n_{1,HIGH}) \times U_{i,1}$	$(n_{2,LOW}, n_{2,HIGH}) \times U_{i,2}$	X	Welfare	CA
1	—	—	$(2,2)\times 2.0$	0	8.0	0
2	—	$(1,-)\times 1.3$	$(1,2)\times 2.3$	1	8.2	1
3	—	$(2,-)\times 1.6$	$(-,2)\times 2.6$	2	8.4	0
4	$(1,-)\times 0.6$	—	$(1,2)\times 2.6$	2	8.4	1
5	$(1,-)\times 0.9$	$(1,-)\times 1.9$	$(-,2)\times 2.9$	3	8.6	1
6	$(2,-)\times 1.2$	—	$(-,2)\times 3.2$	4	8.8	0
7	—	$(-,1)\times 1.9$	$(2,1)\times 2.9$	3	10.6	1
8	—	$(1,1)\times 2.2$	$(1,1)\times 3.2$	4	10.8	3
9	—	$(2,1)\times 2.5$	$(-,1)\times 3.5$	5	11.0	1
10	$(1,-)\times 1.5$	$(-,1)\times 2.5$	$(1,1)\times 3.5$	5	11.0	3
11	$(1,-)\times 1.8$	$(1,1)\times 2.8$	$(-,1)\times 3.8$	6	11.2	3
12	$(2,-)\times 2.1$	$(-,1)\times 3.1$	$(-,1)\times 4.1$	7	11.4	1
13	—	$(-,2)\times 2.8$	$(2,-)\times 3.8$	6	13.2	0
14	$(-,1)\times 1.8$	—	$(2,1)\times 3.8$	6	13.2	1
15	—	$(1,2)\times 3.1$	$(1,-)\times 4.1$	7	13.4	1
16	$(-,1)\times 2.1$	$(1,-)\times 3.1$	$(1,1)\times 4.1$	7	13.4	3
17	—	$(2,2)\times 3.4$	—	8	13.6	0
18	$(-,1)\times 2.4$	$(2,-)\times 3.4$	$(-,1)\times 4.4$	8	13.6	1
19	$(1,-)\times 2.4$	$(-,2)\times 3.4$	$(1,-)\times 4.4$	8	13.6	1
20	$(1,1)\times 2.4$	—	$(1,1)\times 4.4$	8	13.6	3
21	$(1,-)\times 2.7$	$(1,2)\times 3.7$	—	9	13.8	1
22	$(1,1)\times 2.7$	$(1,-)\times 3.7$	$(-,1)\times 4.7$	9	13.8	3
23	$(2,-)\times 3.0$	$(-,2)\times 4.0$	—	10	14.0	0
24	$\mathbf{(2,1)\times 3.0}$	—	$\mathbf{(-,1)\times 5.0}$	**10**	**14.0**	**1**
25	$(-,1)\times 2.7$	$(-,1)\times 3.7$	$(2,-)\times 4.7$	9	15.8	1
26	$(-,1)\times 3.0$	$(1,1)\times 4.0$	$(1,-)\times 5.0$	10	16.0	3
27	$(-,1)\times 3.3$	$(2,1)\times 4.3$	—	11	16.2	1
28	$(1,1)\times 3.3$	$(-,1)\times 4.3$	$(1,-)\times 5.3$	11	16.2	3
29	$(1,1)\times 3.6$	$(1,1)\times 4.6$	—	12	16.4	3
30	$\mathbf{(2,1)\times 3.9}$	$\mathbf{(-,1)\times 4.9}$	—	**13**	**16.6**	**1**
31	$\mathbf{(-,2)\times 3.6}$	—	$\mathbf{(2,-)\times 5.6}$	**12**	**18.4**	**0**
32	$\mathbf{(-,2)\times 3.9}$	$\mathbf{(1,-)\times 4.9}$	$\mathbf{(1,-)\times 5.9}$	**13**	**18.6**	**1**
33	$\mathbf{(-,2)\times 4.2}$	$\mathbf{(2,-)\times 5.2}$	—	**14**	**18.8**	**0**
34	$\mathbf{(1,2)\times 4.2}$	—	$\mathbf{(1,-)\times 6.2}$	**14**	**18.8**	**1**
35	$\mathbf{(1,2)\times 4.5}$	$\mathbf{(1,-)\times 5.5}$	—	**15**	**19.0**	**1**
36	$\mathbf{(2,2)\times 4.8}$	—	—	**16**	**19.2**	**0**

Note: *Allocation denotes the number of allocation, $n_{0,LOW}$ ($n_{1,LOW}$, $n_{2,LOW}$) denotes the number of agents who have a low $MPCR = 0.3$, and choose PC_0 (PC_1, PC_2), respectively, $n_{0,HIGH}$ ($n_{1,HIGH}$, $n_{2,HIGH}$) denotes the number of agents who have a high $MPCR = 0.9$, and choose PC_0 (PC_1, PC_2), respectively, where constraints $n_{LOW} = 2 = n_{0,LOW} + n_{1,LOW} + n_{2,LOW}$, $n_{HIGH} = 2 = n_{0,HIGH} + n_{1,HIGH} + n_{2,HIGH}$ and $n = 4 = n_{LOW} + n_{HIGH}$ are fulfilled and $U_{i,0}$, ($U_{i,1}$, $U_{i,2}$) denotes the individual payoff, which any agent who has selected PC_0 (PC_1, PC_2) will receive, respectively, $(n_{0,LOW}, n_{0,HIGH}) \times U_{i,0}$ ($(n_{1,LOW}, n_{1,HIGH}) \times U_{i,1}$, $(n_{2,LOW}, n_{2,HIGH}) \times U_{i,2}$) denotes total payoff of the group of these agents, respectively (results not displayed), X denotes the total quantity of the public good, Welfare denotes overall payoff of the whole group, CA denotes the number of clone allocations, and allocations in bold denote Pareto-optimal allocations.*

these values show up in Table 8, with 2.0 as individual payoff of a non-contributor in allocation 1 and 4.8 as individual payoff of a full-contributor in allocation 36. With $n = 4$, $m = 3$, $W_{max} = 19.2$, $W_{min} = 8$, $\alpha = 1$, $\beta = 0.3$, $\zeta_{min} = 1$ according to (70), and $\eta_{min} = 1$ according to (71) we may apply the approximation (72) for $NOWL$, which amounts to $9 \leq NOWL \leq 57$, and counting the number of welfare levels in Table 8 confirms this with $NOWL = 25$.

With respect to (73) the individualized benchmark is $\widehat{U}_{i,0} = 4.8$, $\forall i \in \Im_4$, that is already known from the right hand side of the individualized sufficient condition for a prisoner's dilemma situation. For each agent the minimum quantity of the public good that needs to be exceeded in order to achieve Pareto-optimality is $X_{min} \approx 9.333$ according to (76). Applying (77) we conclude that allocations with $X \leq 9$ are not Pareto-optimal, that are allocations 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22 and 25, and Table 8 confirms this result.

According to (78) the truncation constraint for the algorithm is $f_0^1 + 3f_0^2 > 1.4/0.3 \approx 4.667$, where f_0^1 and f_0^2 is the number of full-contributors endowed with a low and high $MPCR$, respectively. Thus, we may apply the algorithm to allocations 23, 26, 27, 28, and 29, which are not Pareto-optimal. The remaining allocations, which are allocations 24, 30, 31, 32, 33, 34, 35, and 36, are Pareto-optimal, because the truncation constraint holds for all subgroups of agents. In the first scenario we obtain $NOPA = 8$ and, therefore, the Pareto-ratio is $8/36 \approx 0.222$. In the second scenario we have to add five clone allocations, which yields $NOPA = 8 + 5 = 13$ and the Pareto-ratio is $13/81 \approx 0.16$.

Finally, it seems worth mentioning that according to Tan (2008) and Zelmer (2003) heterogeneous $MPCR$'s have a negative impact on average contribution levels. Hence, with bearing the results on heterogeneous incomes in mind, it seems that heterogeneity in general leads to lower average contribution levels in linear public goods games. Yet, further experimental evidence is needed to confirm this claim.

6 Conclusion

The generalized calculation procedure introduced in this paper allows for analyzing virtually all linear public goods games with respect to identifying Pareto-optimal allocations and the contributing behavior of human subjects in both homogeneous and heterogeneous parameter settings. We have demonstrated this by analyzing various published linear public goods experiments.

Among other things, it turned out that in standard linear public goods games with homogeneous parameter settings Pareto-optimality of an allocation depends exclusively on the number of subjects who do not fully contribute their budget (income) to the public good, (n_{max}). Given this one dimensional dependence, Pareto-optimality of an allocation could be identified in real time during a running experiment and, therefore, could be directly communicated to participating human subjects with a view to examine how they react to Pareto-optimality. In fact, this possibility allows for a variety of new experimental designs that investigate the role of Pareto-optimality for human decision making in linear public good games and some suggestions have been made in preceding sections.

Moreover, we have demonstrated that the tables on which the calculation procedure for Pareto-optimality rests, may also be used for analyzing the contribution behavior of human subjects. One such example is the contribution behavior of the rich and poor in heterogeneous income settings. In particular, we have shown that the poor may have a much higher chance to benefit from contributing to the public good than the rich, which in turn may determine their actual contributing behavior. By analyzing the results of two published experiments with heterogeneous income settings we found some evidence supporting this hypothesis. But again, new experimental designs may now be developed to test this and other hypothesis.

To conclude, the calculation procedures introduced in this paper are a useful tool for anyone designing a linear public goods experiment and may, therefore, be helpful to better understand human subject behavior in such environments.

A Appendix

In the following, we derive the approximation of $NOPA$ according to (53) and present Table A1, which contains all model parameters and definitions. In addition, we include the MATLAB version of the calculation procedure, which allows for reproducing practically all tables shown in this paper and which may be used for calculating NOA, $NOWL$, $NOPA$, etc. for all permissible parameter values.

To begin with we examine two cases for the approximation of $NOPA$ that coincides with the number of lattice points G.

First case: Assumption $n - \frac{NOWL_3-1}{2} \in \mathbb{N}$

$$F = \frac{1}{2} \cdot (NOWL_3 - 1) \cdot \frac{NOWL_3 - 1}{2} \tag{A.1}$$

$$\hat{G} = 2 \cdot (NOWL_3 - 1) \tag{A.2}$$

$$\begin{aligned} G &= \underbrace{\frac{1}{2} \cdot (NOWL_3 - 1) \cdot \frac{NOWL_3 - 1}{2}}_{F} + \frac{1}{2} \cdot \underbrace{2 \cdot (NOWL_3 - 1)}_{\hat{G}} + 1 \\ &= \left\lfloor \frac{(NOWL_3 - 1)^2}{4} \right\rfloor + NOWL_3 \end{aligned} \tag{A.3}$$

Second case: Assumption $n - \frac{NOWL_3-1}{2} \notin \mathbb{N}$

$$F = \frac{1}{2} \cdot (NOWL_3 - 1) \cdot \frac{NOWL_3 - 1}{2} - \frac{1}{4} \tag{A.4}$$

$$\hat{G} = 2 \cdot (NOWL_3 - 1) \tag{A.5}$$

$$\begin{aligned} G &= \underbrace{\frac{1}{2} \cdot (NOWL_3 - 1) \cdot \frac{NOWL_3 - 1}{2} - \frac{1}{4}}_{F} + \frac{1}{2} \cdot \underbrace{2 \cdot (NOWL_3 - 1)}_{\hat{G}} + 1 \\ &= \left\lfloor \frac{(NOWL_3 - 1)^2}{4} \right\rfloor + NOWL_3 \end{aligned} \tag{A.6}$$

Table A1: List of Abbreviations, Definitions and Model Parameters

Parameter	Description	First Appearance in	Eq.
Abbreviations			
CA	Clon allocation	Section 2.2	
FC	Full contribution	Section 2.2	
H	High potential for Pareto-optimal allocations	Section 5.2	
L	Low potential for Pareto-optimal allocations	Section 5.2	
LR	Weighted loss-ratio	Section 5.5	(81)
LR^p	Loss-ratio of subgroup p	Section 5.4	(79)
$MPCR$	Marginal per capita return	Section 1	(6)
$MPCR_i$	i-th agent's marginal per capita return	Section 4.1	(62)
N	No Pareto-optimal allocations	Section 5.2	
NC	Non-contribution	Section 2.2	
NOA	Number of feasible allocations	Section 2.1	(9)
$NOA(con)^p$	... where at least one agent of subgroup p contributes to the public good	Section 5.4	
$NOA(hi)^p$	... where at least one agent of subgroup p contributes to the public good and these contributing agents get a strictly higher payoff than in Nash equilibrium	Section 5.4	
$NOA(k)$	... with k free-riders	Section 3.1	
$NOA(lo)^p$	... where at least one agent of subgroup p contributes to the public good and these contributing agents get a strictly lower payoff than in Nash equilibrium	Section 5.4	
$NOPA$	Number of Pareto-optimal allocations	Section 1	(11)
$NOWL$	Number of welfare levels	Section 2.1	(10)
$NOWL_m$	... potentially associated with Pareto-optimal allocations	Section 3.2.3	(41)
PC	Partial contribution	Section 2.2	
PO	Pareto-optimality	Section 5.2	
PR	Weighted profit-ratio	Section 5.5	(82)
PR^p	Profit-ratio of subgroup p	Section 5.4	(80)
R_A	Results for allocated income	Section 5.4	
R_E	Results for earned income	Section 5.4	
R_W	Results for wealth	Section 5.4	
Y	One or more than one Pareto-optimal allocations	Section 5.2	
Definitions			
Pareto-ratio	Share of Pareto-optimal allocations ($NOPA$) in the set of feasible allocations (NOA)	Section 2.1	(12)
Sets			
$\mathfrak{A}$	Set of alternatives	Section 2.1	
$\mathfrak{A}_i$	i-th agent's set of alternatives	Section 4.2	
$\mathfrak{A}^p$	p-th subgroup's set of alternatives	Section 4.2.1	
$\mathfrak{D}$	Domain, set of feasible allocations	Section 2.1	
$\widehat{\mathfrak{D}}$	Set coinciding with $\mathfrak{D}$	Section 3.2.2	
$\mathfrak{G}$	Last three relevant items of the normal form	Section 5.4	
$\mathfrak{F}$	Normal form of linear public goods games	Section 2.1	(61)
$\mathfrak{I}_n$	Set of n agents	Section 2.1	
$\mathfrak{I}_m$	Index set of m alternatives	Section 3.2.1	
$\mathbb{N}$	Set of all natural numbers	Section 2.1	
$\mathbb{N}_0$	Set of all natural numbers including zero	Section 2.1	
$\mathbb{N}_0^m$	Set of m-tuples where entries contain natural numbers including zero	Section 3.2.2	
$\mathfrak{P}$	Set of Pareto-optimal allocations	Section 2.1	
$\mathbb{Q}^+$	Set of all positive rational numbers	Section 2.1	
$\mathbb{R}$	Set of all real numbers	Section 3.1	
$\mathfrak{S}_n$	Symmetric group of degree n	Section 3.1	
$\mathfrak{W}$	Image, set of feasible levels of welfare	Section 2.1	
Latin Characters			
B	Budget	Section 2.1	(3)
B_i	i-th agent's budget	Section 4.1	

Table A1: Continued

Parameter	Description	First Appearance in	Eq.
Latin Characters			
B^p	p-th subgroup's budget	Section 4.1	
f^p	p-th subgroup's number of agents	Section 4.1	(60)
f_0^p	p-th subgroup's number of full contributors	Section 4.3.1	(78)
f_j^p	p-th subgroup's number of agents		
	who choose alternative PC_j	Section 4.3.1	
F	Area of a lattice polygon	Section 3.3	(48)
g	Gini coefficient	Section 5.5	
G	Lattice points of a lattice polygon	Section 3.3	(48)
$\hat{G}$	Boundary lattice points of a lattice polygon	Section 3.3	(48)
i	i-th representative agent	Section 2.1	
m	Number of alternatives	Section 2.1	(4)
m_i	i-th agent's number of alternatives	Section 4.1	
m^p	p-th subgroup's number of alternatives	Section 4.1	
n	Number of agents	Section 2.1	
n_0	Number of full contributors	Section 3.2.1	(44)
n_{FC}	Number of agents who choose alternative FC	Section 2.2	(21)
n_j	Number of agents who choose alternative PC_j	Section 3.2.2	
n_{max}	Maximum number of free-riders	Section 3.1	(23)
n_{min}	Minimum number of full contributors	Section 3.1	(22)
n_{NC}	Number of agents who choose alternative NC	Section 2.2	
n_{PC}	Number of agents who choose alternative PC	Section 2.2	
$N_{(k,r)}$	Number of agents who choose alternative PC_k		
	in the r-th step of the algorithm	Section 3.5	
p	p-th representative subgroup of agents	Section 4.1	
p^*	Subgroup of agents endowed with the largest budget	Section 4.2.1	
P	Number of subgroups	Section 4.1	
PC_j	$(j+1)$-th partial contribution alternative	Section 3.2.1	
r	r-th step of the algorithm	Section 3.5	
S	Number of steps of the algorithm until the		
	truncation constraint is fulfilled	Section 3.5	
TB	Total budget	Section 5.5	
U	Payoff function	Section 2.1	(2)
$\widehat{U}_0$	Generalized benchmark	Section 3.2.3	(34)
$\widehat{U}_{i,0}$	i-th agent's individualized benchmark	Section 4.3.1	(73)
$\widehat{U}_{FC}$	Binary decision space benchmark	Section 3.1	(19)
U_0	Individual payoff of full contributors	Section 3.2.2	(28)
U_i	i-th agent's payoff function	Section 3.1	(15)
U_{FC}	Individual payoff if alternative FC is selected	Section 2.2	(15)
U_k	Individual payoff if alternative PC_k is selected	Section 3.2.2	(29)
U_{m-1}	Individual payoff of non-contributors	Section 3.2.2	(30)
U_{NC}	Individual payoff if alternative NC is selected	Section 2.2	(16)
U_{PC}	Individual payoff if alternative PC is selected	Section 2.2	
W	Welfare	Section 2.1	(8)
$W(n_{FC})$	Welfare binary decision space	Section 3.1	(8)
$W(n_0, \dots, n_{m-2})$	Welfare multiple decision space	Section 3.2.3	(31)
$W(x_1, \dots, x_n)$	Generalized welfare	Section 4.1	(63)
W^{-1}	Inverse of correspondence W	Section 3.2.3	(42)
W_{max}	Maximum level of welfare	Section 3.1	
W_{min}	Minimum level of welfare	Section 3.1	
x_i	i-th agent's contribution to the public good	Section 2.1	
X	Total quantity of the public good	Section 2.1	(1)
$X_{(r)}$	Public good in step r of the algorithm	Section 3.5	
$X_{(r)}^p$	...with respect to the p-th subgroup	Section 4.3.1	
X_i	i-th agent's consumption of the public good	Section 2.1	
X_{min}	Minimum quantity of the public good which		
	must be exceeded to allow for Pareto-optimality	Section 3.2.3	(37)
X_{min}^p	...with respect to the p-th subgroup	Section 4.3.1	(76)
y_i	i-th agent's quantity of the private good	Section 2.1	

Table A1: Continued

Parameter	Description	First Appearance in	Eq.
Greek Characters			
α	Payoff multiplier private good	Section 2.1	(2)
α_i	i-th agent's payoff multiplier private good	Section 4.1	
α^p	p-th subgroup's payoff multiplier private good	Section 4.1	
β	Payoff multiplier public good	Section 2.1	(2)
β_i	i-th agent's payoff multiplier public good	Section 4.1	
β^p	p-th subgroup's payoff multiplier public good	Section 4.1	
γ	Budget multiplier private good	Section 2.1	(3)
γ_i	i-th agent's budget multiplier private good	Section 4.1	
γ^p	p-th subgroup's budget multiplier private good	Section 4.1	
δ	Budget multiplier public good	Section 2.1	(3)
δ_i	i-th agent's budget multiplier public good	Section 4.1	
δ^p	p-th subgroup's budget multiplier public good	Section 4.1	
ϵ	Smallest possible unit in which the budget may be spend	Section 2.1	(4)
ϵ_i	i-th agent's smallest possible unit in which the budget may be spend	Section 4.1	
ϵ^p	p-th subgroup's smallest possible unit in which the budget may be spend	Section 4.1	
ζ	Smallest possible unit in which the private good may be produced	Section 2.1	(5)
ζ_i	i-th agent's smallest possible unit in which the private good may be produced	Section 4.1	
ζ_{min}	Minimum of the smallest possible unit in which the private good may be produced	Section 4.3	(70)
ζ^p	p-th subgroup's smallest possible unit in which the private good may be produced	Section 4.1	
η	Smallest possible unit in which the public good may be produced	Section 2.1	(5)
η_i	i-th agent's smallest possible unit in which the public good may be produced	Section 4.1	
η_{min}	Minimum of the smallest possible unit in which the public good may be produced	Section 4.3	(71)
η^p	p-th subgroup's smallest possible unit in which the public good may be produced	Section 4.1	
ι	Identity permutation	Section 3.1	
$\pi_{i,\rho}$	i-th agent's probability with respect to the ρ-th alternative	Section 2.2	
σ	Permutation binary decision space	Section 3.1	
τ	Permutation multiple decision space	Section 3.2.2	
Symbols			
$\subset$	Subset	Section 2.1	
$\in$	Element of a set	Section 2.1	
$\notin$	No element of a set	Section 3.3	
$\forall$	For all	Section 2.1	
$\sum$	Sum	Section 2.1	
$\times$	Multiplication	Section 2.2	
!	Factorial	Section 2.2	
$\prod$	Product	Section 3.4	
$\|\mathfrak{A}\|$	Number of elements of set $\mathfrak{A}$	Section 2.1	
$\lfloor \ldots \rfloor$	Floor function	Section 3.1	(53)
$\lceil \ldots \rceil$	Ceiling function	Section 3.1	(21)
$[\ldots]$	Results for the second scenario	Section 5.2	
Δ_1	Triangle given by the n_0-axis, n_1-axis and group size constraint	Section 3.3	
Δ_2	Triangle given by the n_0-axis, group size constraint and $NOPA$ approximation constraint	Section 3.3	
Δ_3	Triangle given by the n_0-axis, n_{min}-line and group size constraint	Section 3.3	

MATLAB Code

The code is setup in MATLAB version 7.8.0. R2009a / R2010a and allows for reproducing almost all tables shown in this paper. To illustrate the code, we use the case shown in Table 4. Recall that in general we denote the heterogeneous income setting with two budgets, $P = 2$, where agents are indistinguishable as $\mathfrak{F} = \{(\alpha^1; \beta^1; \gamma^1; \delta^1; B^1; m^1; f^1), (\alpha^2; \beta^2; \gamma^2; \delta^2; B^2; m^2; f^2)\}$, that is, the first scenario. With respect to the parameter set of Table 4 we obtain $\mathfrak{F} = \{(1; 0.5; 1; 1; 1; 2; 2), (1; 0.5; 1; 1; 2; 3; 2)\}$, which implies $\epsilon = 1$. To apply the code, start MATLAB and load the PDF of this paper. Next, copy the entire code shown below and paste it into the command window of MATLAB and press 'enter' to run the code. The output of the code is displayed in the workspace window and shows Table 4 and related results shown in lines seven to nine of Table 6.

Inspection of the code shows that the calculation procedures are solved step wise. The variable *step* indicates the progress of calculus and offers the opportunity of saving values of some auxiliary variables, so that the code may be restarted from the advanced status to avoid out of memory effects. In the $P = 2$ case the code first derives two numbers of alternatives, separately for each group of agents endowed with the same budget. Second, the code generates two basic tables, again separately for each group of agents, containing all feasible numbers of agents who choose an alternative. Third, column size of the basic tables is automatically read by the code. Fourth, basic tables are extended and combined to obtain a matching table containing all feasible numbers of agents who choose an alternative. Fifth, row size of the matching table is automatically read by the code, which is the NOA, and some auxiliary variables are introduced. Sixth, the individual contribution to the public good and the total quantity of the public good X is calculated for each allocation. Seventh, utility obtained from nonrival consumption of the public good and utility, according to (2), is derived. Eighth, welfare is calculated, according to (63). Ninth, Pickhardt's Table is composed in the same manner as Table 1 or Table 2, but without an 'Allocation' and CA column. Note that the order of allocations may differ between Pickhardt's Table generated with the code and tables shown in this paper. Tenth, results for NOA, $NOWL$, n_{max}, n_{min}, $NOPA$ and the Pareto-ratio are provided by the code. Eleventh, the code calculates for each subgroup of agents with the same budget, a loss-ratio (profit-ratio), which we have defined in section 5.4 and, the weighted loss-ratio (profit-ratio), which we have defined in section 5.5. In general the range of these steps is increasing while raising P, for instance the second step requires generating P basic tables.

Also, parameters B^1, f^1, B^2, f^2, B^3, f^3, B^4, f^4, B^5, f^5, α, β and ϵ, are denoted in the code as B01, f01, B02, f02, B03, f03, B04, f04, B05, f05, alpha, beta and epsilon, respectively. By setting $B^2 = f^2 = B^3 = f^3 = B^4 = f^4 = B^5 = f^5 = 0$ the code can also be used for $P = 1$, that is, the first scenario in homogeneous parameter settings. For instance, to generate Table 1, the input parameter values must be changed to: $B^1 = 2$, $f^1 = 5$, $B^2 = f^2 = B^3 = f^3 = B^4 = f^4 = B^5 = f^5 = 0$, $\alpha = 4, \beta = 1$ and $\epsilon = 1$, which in code language translate to: B01=2; f01=5; B02=0; f02=0; B03=0; f03=0; B04=0; f04=0; B05=0; f05=0; alpha = 4; beta = 1; epsilon = 1.

Finally, the code may be used for any permissible parameter setting, subject to $P \leq 5$, $\gamma = \delta = 1$ and the constraints mentioned in the main text. Moreover, it is possible, by modifying α and β, to adapt the code for homogeneous parameter settings and heterogeneous income cases, where $\gamma = \delta = 1$ does not hold. Further, the code may be developed for numerically specified cases with heterogeneous $MPCR$s (productivities) and for $P > 5$.

```
%Begin Code
clear all
%Step 0: Input
B01=1; f01=2; B02=2; f02=2; B03=0; f03=0; B04=0; f04=0; B05=0; f05=0;
alpha = 1; beta = 0.5; epsilon = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 1: Number of alternatives
step=1
if f01==0 m01=0;
else m01=round(B01/epsilon)+1;
end
if f02==0 m02=0;
else m02=round(B02/epsilon)+1;
end
if f03==0 m03=0;
else m03=round(B03/epsilon)+1;
end
if f04==0 m04=0;
else m04=round(B04/epsilon)+1;
end
if f05==0 m05=0;
else m05=round(B05/epsilon)+1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 2: Generating basic tables
step=2
%Generating first basic table denoted as T01
A=[0:f01]';
[M,N]=size(A);
while N < m01
for k=0:f01
if k==0 C=zeros(M,1);
else B=k*ones(M,1); C=[C;B];
end
end
for k=0:f01
if k==0 D=A;
else D=[A;D];
end
end
E=[D C];
clear A B C D
[K,L]=size(E);
h=1
for j=1:K
if sum(E(j,:)) < f01+1 F(h,:)=E(j,:); h=h+1
else
end
end
A=F;
clear E F
[M,N]=size(A);
end
h=1
for k=1:M
if sum(A(k,:))==f01 T01(h,:)=A(k,:); h=h+1
end
end
clear A K L M N
step=2.0001
%Generating second basic table denoted as T02
A=[0:f02]';
[M,N]=size(A);
while N < m02
for k=0:f02
if k==0 C=zeros(M,1);
else B=k*ones(M,1); C=[C;B];
end
end
for k=0:f02
if k==0 D=A;
else D=[A;D];
end
end
E=[D C];
clear A B C D
[K,L]=size(E);
h=1
for j=1:K
if sum(E(j,:)) < f02+1 F(h,:)=E(j,:); h=h+1
else
end
end
A=F;
clear E F
[M,N]=size(A);
end
h=1
for k=1:M
if sum(A(k,:))==f02 T02(h,:)=A(k,:); h=h+1
end
end
clear A K L M N
step=2.0002
%Generating third basic table denoted as T03
A=[0:f03]';
[M,N]=size(A);
while N < m03
for k=0:f03
```

```
if k==0 C=zeros(M,1);
else B=k*ones(M,1); C=[C;B];
end
end
for k=0:f03
if k==0 D=A;
else D=[A;D];
end
end
E=[D C];
clear A B C D
[K,L]=size(E);
h=1
for j=1:K
if sum(E(j,:)) < f03+1 F(h,:)=E(j,:); h=h+1
else
end
end
A=F;
clear E F
[M,N]=size(A);
end
h=1
for k=1:M
if sum(A(k,:))==f03 T03(h,:)=A(k,:); h=h+1
end
end
clear A K L M N
step=2.0003
%Generating fourth basic table denoted as T04
A=[0:f04]';
[M,N]=size(A);
while N < m04
for k=0:f04
if k==0 C=zeros(M,1);
else B=k*ones(M,1); C=[C;B];
end
end
for k=0:f04
if k==0 D=A;
else D=[A;D];
end
end
E=[D C];
clear A B C D
[K,L]=size(E);
h=1
for j=1:K
if sum(E(j,:)) < f04+1 F(h,:)=E(j,:); h=h+1
else
end
end
A=F;
clear E F
[M,N]=size(A);
end
h=1
for k=1:M
if sum(A(k,:))==f04 T04(h,:)=A(k,:); h=h+1
end
end
clear A K L M N
step=2.0004
%Generating fifth basic table denoted as T05
A=[0:f05]';
[M,N]=size(A);
while N < m05
for k=0:f05
if k==0 C=zeros(M,1);
else B=k*ones(M,1); C=[C;B];
end
end
for k=0:f05
if k==0 D=A;
else D=[A;D];
end
end
E=[D C];
clear A B C D
[K,L]=size(E);
h=1
for j=1:K
if sum(E(j,:)) < f05+1 F(h,:)=E(j,:); h=h+1
else
end
end
A=F;
clear E F
[M,N]=size(A);
end
h=1
for k=1:M
if sum(A(k,:))==f05 T05(h,:)=A(k,:); h=h+1
end
end
clear A K L M N
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 3: Column size of basic tables
step=3
```

```
sizeT01=size(T01,1);
sizeT02=size(T02,1);
sizeT03=size(T03,1);
sizeT04=size(T04,1);
sizeT05=size(T05,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 4: Matching basic tables
step=4
B=ones(sizeT04,1);
for j=1:sizeT05
if j==1 Help01=B*T05(1,:);
else A=B*T05(j,:); Help01=[Help01;A]; clear A
j
end
end
clear B
step=4.0001
for j=1:sizeT05
if j==1 Help02=T04;
else Help02=[Help02;T04]; j
end
end
step=4.0002
Help03=[Help02 Help01];
step=4.0003
B=ones(sizeT03,1);
for j=1:(sizeT05*sizeT04)
if j==1 Help04=B*Help03(1,:);
else A=B*Help03(j,:); Help04=[Help04;A]; clear A
j
end
end
clear B
step=4.0004
for j=1:(sizeT05*sizeT04)
if j==1 Help05=T03;
else Help05=[Help05;T03]; j
end
end
step=4.0005
Help06=[Help05 Help04];
step=4.0006
B=ones(sizeT02,1);
for j=1:(sizeT05*sizeT04*sizeT03)
if j==1 Help07=B*Help06(1,:);
else A=B*Help06(j,:); Help07=[Help07;A]; clear A
j
end
end
clear B
step=4.0007
for j=1:(sizeT05*sizeT04*sizeT03)
if j==1 Help08=T02;
else Help08=[Help08;T02]; j
end
end
step=4.0008
Help09=[Help08 Help07];
step=4.0009
B=ones(sizeT01,1);
for j=1:(sizeT05*sizeT04*sizeT03*sizeT02)
if j==1 Help10=B*Help09(1,:);
else A=B*Help09(j,:); Help10=[Help10;A]; clear A
j
end
end
clear B
step=4.0010
for j=1:(sizeT05*sizeT04*sizeT03*sizeT02)
if j==1 Help11=T01;
else Help11=[Help11;T01]; j
end
end
step=4.0011
Help12=[Help11 Help10];
step=4.0012
%Matching Table
MT=Help12;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 5: Size of tables and auxiliary variables
step=5
sizeMT=size(MT,1);
sizeT01PG=size(T01,2);
sizeT02PG=size(T02,2);
sizeT03PG=size(T03,2);
sizeT04PG=size(T04,2);
sizeT05PG=size(T05,2);
sizePG=[sizeT01PG sizeT02PG sizeT03PG sizeT04PG sizeT05PG];
cumsizePG=cumsum(sizePG,2);
PGHelp1=MT(1:sizeMT,1:cumsizePG(1));
PGHelp2=MT(1:sizeMT,1+cumsizePG(1):cumsizePG(2));
PGHelp3=MT(1:sizeMT,1+cumsizePG(2):cumsizePG(3));
PGHelp4=MT(1:sizeMT,1+cumsizePG(3):cumsizePG(4));
PGHelp5=MT(1:sizeMT,1+cumsizePG(4):cumsizePG(5));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 6: Public good
step=6
PG1=zeros(sizeMT,sizePG(1));
for j=1:sizePG(1)
```

```
PG1(:,j)=PGHelp1(:,j)*(sizePG(1)-j); j
end
step=6.0001
PG2=zeros(sizeMT,sizePG(2));
for j=1:sizePG(2)
PG2(:,j)=PGHelp2(:,j)*(sizePG(2)-j); j
end
step=6.0002
PG3=zeros(sizeMT,sizePG(3));
for j=1:sizePG(3)
PG3(:,j)=PGHelp3(:,j)*(sizePG(3)-j); j
end
step=6.0003
PG4=zeros(sizeMT,sizePG(4));
for j=1:sizePG(4)
PG4(:,j)=PGHelp4(:,j)*(sizePG(4)-j); j
end
step=6.0004
PG5=zeros(sizeMT,sizePG(5));
for j=1:sizePG(5)
PG5(:,j)=PGHelp5(:,j)*(sizePG(5)-j); j
end
step=6.0005
PGContribution=[PG1 PG2 PG3 PG4 PG5];
step=6.0006
X=sum(PGContribution')'*epsilon;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 7: Utility
step=7
UtilityPG=X*beta;
step=7.0001
U1=ones(sizeMT,sizePG(1))*epsilon;
for j=1:sizePG(1)
U1(:,j)=U1(:,j)*(j-1); j
end
step=7.0002
U2=ones(sizeMT,sizePG(2))*epsilon;
for j=1:sizePG(2)
U2(:,j)=U2(:,j)*(j-1); j
end
step=7.0003
U3=ones(sizeMT,sizePG(3))*epsilon;
for j=1:sizePG(3)
U3(:,j)=U3(:,j)*(j-1); j
end
step=7.0004
U4=ones(sizeMT,sizePG(4))*epsilon;
for j=1:sizePG(4)
U4(:,j)=U4(:,j)*(j-1); j
end
step=7.0005
U5=ones(sizeMT,sizePG(5))*epsilon;
for j=1:sizePG(5)
U5(:,j)=U5(:,j)*(j-1); j
end
step=7.0006
UT1=zeros(sizeMT,sizePG(1));
for j=1:sizePG(1)
UT1(:,j)=U1(:,j)*alpha+UtilityPG; j
end
step=7.0007
UT2=zeros(sizeMT,sizePG(2));
for j=1:sizePG(2)
UT2(:,j)=U2(:,j)*alpha+UtilityPG; j
end
step=7.0008
UT3=zeros(sizeMT,sizePG(3));
for j=1:sizePG(3)
UT3(:,j)=U3(:,j)*alpha+UtilityPG; j
end
step=7.0009
UT4=zeros(sizeMT,sizePG(4));
for j=1:sizePG(4)
UT4(:,j)=U4(:,j)*alpha+UtilityPG; j
end
step=7.0010
UT5=zeros(sizeMT,sizePG(5));
for j=1:sizePG(5)
UT5(:,j)=U5(:,j)*alpha+UtilityPG; j
end
step=7.0011
U=[UT1 UT2 UT3 UT4 UT5];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 8: Welfare
step=8
IndividualWelfare1=PGHelp1.*UT1;
for j=1:sizeMT
for k=1:sizePG(1)
if IndividualWelfare1(j,k)==0
else IndividualWelfare1(j,k)=UT1(j,k);
end
j
k
end
end
step=8.0001
IndividualWelfare2=PGHelp2.*UT2;
for j=1:sizeMT
for k=1:sizePG(2)
```

```
if IndividualWelfare2(j,k)==0
else IndividualWelfare2(j,k)=UT2(j,k);
end
j
k
end
end
step=8.0002
IndividualWelfare3=PGHelp3.*UT3;
for j=1:sizeMT
for k=1:sizePG(3)
if IndividualWelfare3(j,k)==0
else IndividualWelfare3(j,k)=UT3(j,k);
end
j
k
end
end
step=8.0003
IndividualWelfare4=PGHelp4.*UT4;
for j=1:sizeMT
for k=1:sizePG(4)
if IndividualWelfare4(j,k)==0
else IndividualWelfare4(j,k)=UT4(j,k);
end
j
k
end
end
step=8.0004
IndividualWelfare5=PGHelp5.*UT5;
for j=1:sizeMT
for k=1:sizePG(5)
if IndividualWelfare5(j,k)==0
else IndividualWelfare5(j,k)=UT5(j,k);
end
j
k
end
end
step=8.0005
GroupWelfare=MT.*U;
step=8.0006
Welfare=sum(GroupWelfare')';
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 9: PickhardtsTable
step=9
Table1=zeros(sizeMT,sizePG(1)*2);
h=1;
k=1;
for j=1:sizePG(1)*2
if mod(j,2)==1 Table1(:,j)=PGHelp1(:,h); h=h+1;
else Table1(:,j)=IndividualWelfare1(:,k); k=k+1;
end
j
end
step=9.0001
Table2=zeros(sizeMT,sizePG(2)*2);
h=1;
k=1;
for j=1:sizePG(2)*2
if mod(j,2)==1 Table2(:,j)=PGHelp2(:,h); h=h+1;
else Table2(:,j)=IndividualWelfare2(:,k); k=k+1;
end
j
end
step=9.0002
Table3=zeros(sizeMT,sizePG(3)*2);
h=1;
k=1;
for j=1:sizePG(3)*2
if mod(j,2)==1 Table3(:,j)=PGHelp3(:,h); h=h+1;
else Table3(:,j)=IndividualWelfare3(:,k); k=k+1;
end
j
end
step=9.0003
Table4=zeros(sizeMT,sizePG(4)*2);
h=1;
k=1;
for j=1:sizePG(4)*2
if mod(j,2)==1 Table4(:,j)=PGHelp4(:,h); h=h+1;
else Table4(:,j)=IndividualWelfare4(:,k); k=k+1;
end
j
end
step=9.0004
Table5=zeros(sizeMT,sizePG(5)*2);
h=1;
k=1;
for j=1:sizePG(5)*2
if mod(j,2)==1 Table5(:,j)=PGHelp5(:,h); h=h+1;
else Table5(:,j)=IndividualWelfare5(:,k); k=k+1;
end
j
end
step=9.0005
PickhardtsTable=[Table1 Table2 Table3 Table4 Table5 X Welfare];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Step 10: Output NOA, NOWL, ...
step=10
NOA=sizeMT;
d=Welfare(1)-Welfare(2);
NOWL=1+(Welfare(1)-Welfare(sizeMT))/d;
FC=Table1(:,1)+Table2(:,1)+Table3(:,1)+Table4(:,1)+Table5(:,1);
nmax=ceil(alpha/beta)-1;
n=f01+f02+f03+f04+f05;
nmin=n-nmax;
Condition=(FC > nmin-1);
NOPA=sum(Condition);
ParetoRatio=NOPA/NOA;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Step 11: Output Profit and Loss-ratios
step=11
T=f01*B01+f02*B02+f03*B03+f04*B04+f05*B05;
IS01=IndividualWelfare1(:,1);
h=2;
while h<sizeT01PG
IS01=[IS01 IndividualWelfare1(:,h)];
h=h+1;
end
Benchmark01=IndividualWelfare1(sizeMT,sizeT01PG);
IS01C1=(IS01>0 & IS01<Benchmark01);
IS01C2=(sum(IS01C1,2)>0);
IS01C3=(IS01>0 & IS01<= Benchmark01);
IS01C4=(sum(IS01C3,2)>0);
IS01C5=(IS01==0);
IS01C6=(sum(IS01C5,2)==sizeT01PG-1);
LossRatio01=sum(IS01C2)/(NOA-sum(IS01C6));
ProfitRatio01=1-sum(IS01C4)/(NOA-sum(IS01C6));
step=11.0001
IS02=IndividualWelfare2(:,1);
h=2;
while h<sizeT02PG
IS02=[IS02 IndividualWelfare2(:,h)];
h=h+1;
end
Benchmark02=IndividualWelfare2(sizeMT,sizeT02PG);
IS02C1=(IS02>0 & IS02<Benchmark02);
IS02C2=(sum(IS02C1,2)>0);
IS02C3=(IS02>0 & IS02<=Benchmark02);
IS02C4=(sum(IS02C3,2)>0);
IS02C5=(IS02==0);
IS02C6=(sum(IS02C5,2)==sizeT02PG-1);
LossRatio02=sum(IS02C2)/(NOA-sum(IS02C6));
ProfitRatio02=1-sum(IS02C4)/(NOA-sum(IS02C6));
step=11.0002
IS03=IndividualWelfare3(:,1);
h=2;
while h<sizeT03PG
IS03=[IS03 IndividualWelfare3(:,h)];
h=h+1;
end
Benchmark03=IndividualWelfare3(sizeMT,sizeT03PG);
IS03C1=(IS03>0 & IS03<Benchmark03);
IS03C2=(sum(IS03C1,2)>0);
IS03C3=(IS03>0 & IS03<=Benchmark03);
IS03C4=(sum(IS03C3,2)>0);
IS03C5=(IS03==0);
IS03C6=(sum(IS03C5,2)==sizeT03PG-1);
LossRatio03=sum(IS03C2)/(NOA-sum(IS03C6));
ProfitRatio03=1-sum(IS03C4)/(NOA-sum(IS03C6));
step=11.0003
IS04=IndividualWelfare4(:,1);
h=2;
while h<sizeT04PG
IS04=[IS04 IndividualWelfare4(:,h)];
h=h+1;
end
Benchmark04=IndividualWelfare4(sizeMT,sizeT04PG);
IS04C1=(IS04>0 & IS04<Benchmark04);
IS04C2=(sum(IS04C1,2)>0);
IS04C3=(IS04>0 & IS04<=Benchmark04);
IS04C4=(sum(IS04C3,2)>0);
IS04C5=(IS04==0);
IS04C6=(sum(IS04C5,2)==sizeT04PG-1);
LossRatio04=sum(IS04C2)/(NOA-sum(IS04C6));
ProfitRatio04=1-sum(IS04C4)/(NOA-sum(IS04C6));
step=11.0004
IS05=IndividualWelfare5(:,1);
h=2;
while h<sizeT05PG
IS05=[IS05 IndividualWelfare5(:,h)];
h=h+1;
end
Benchmark05=IndividualWelfare5(sizeMT,sizeT05PG);
IS05C1=(IS05>0 & IS05<Benchmark05);
IS05C2=(sum(IS05C1,2)>0);
IS05C3=(IS05>0 & IS05<=Benchmark05);
IS05C4=(sum(IS05C3,2)>0);
IS05C5=(IS05==0);
IS05C6=(sum(IS05C5,2)==sizeT05PG-1);
LossRatio05=sum(IS05C2)/(NOA-sum(IS05C6));
ProfitRatio05=1-sum(IS05C4)/(NOA-sum(IS05C6));
step=11.0005
ProfitRatio=(ProfitRatio01*f01*B01+ProfitRatio02*f02*B02+ProfitRatio03*f03*B03+ProfitRatio04*f04*B04+ProfitRatio05*f05*B05)/T;
LossRatio=(LossRatio01*f01*B01+LossRatio02*f02*B02+LossRatio03*f03*B03+LossRatio04*f04*B04+LossRatio05*f05*B05)/T;
%End Code
```

References

Andreoni, J. (1995). Cooperation in Public Goods Experiments: Kindness or Confusion?, *American Economic Review*, 85(4), pp. 891-904.

Andreoni, J. and Petrie R. (2004). Public good experiments without confidentiality: a glimpse into fund-raising, *Journal of Public Economics*, 88, pp. 1605-1623.

Bardsley, N. and Sausgruber, R. (2005). Conformity and Reciprocity in Public Goods Provision, *Journal of Economic Psychology*, 26, pp. 664-681.

Bazart, C. and Pickhardt, M. (2011). Fighting Income Tax Evasion with Positive Rewards, *Public Finance Review*, 39(1), pp. 124-149.

Bergstrom, T., Blume, L. and Varian, H. (1986). On the Private Provision of Public Goods, *Journal of Public Economics*, 29, pp. 25-49.

Brandts, J. and Schram, A. (2001). Cooperation and noise in public goods experiments: applying the contribution function approach, *Journal of Public Economics*, 79, pp. 399-427.

Brown-Kruse, J. and Hummels, D. (1993). Gender effects in laboratory public goods contribution, *Journal of Economic Behavior and Organization*, 22, pp. 255-267.

Buckley, E. and Croson, R. (2006). Income and wealth heterogeneity in the voluntary provision of linear public goods, *Journal of Public Economics*, 90, pp. 935-955.

Burlando, R. M. and Guala, F. (2005). Heterogeneous Agents in Public Goods Experiments, *Experimental Economics*, 8, pp. 35-54.

Carpenter, J. P. (2004). When in Rome: conformity and the provision of public goods, *Journal of Socio-Economics*, 33, pp. 395-408.

Chaudhuri, A. (2011). Sustaining cooperation in laboratory public goods experiments: a selective survey of the literature, *Experimental Economics*, 14(1), pp. 47-83.

Cherry, T. L., Kroll, S. and Shogren, J. F. (2005). The impact of endowment heterogeneity and origin on public good contributions: evidence from the lab, *Journal of Economic Behavior and Organization*, 57, pp. 357-365.

Cinyabuguma, M., Page, T. and Putterman, L. (2005). Cooperation under the threat of expulsion in a public goods experiment, *Journal of Public Economics*, 89, pp. 1421-1435.

Cox, J. and Sadiraj, V. (2007). On Modeling Voluntary Contributions to Public Goods, *Public Finance Review*, 35(2), pp. 311-332.

Croson, R. T. A. (2007). Theories of Commitment, Altruism and Reciprocity: Evidence from Linear Public Goods Games, *Economic Inquiry*, 45, pp. 199-216.

De Loera, J. A., Hemmecke R., Tauzer, J. and Yoshida, R. (2004). Effective lattice point counting in rational convex polytopes, *Journal of Symbolic Computation*, 38, pp. 1273-1302.

Denant-Boemont, L., Masclet, D. and Noussair, C. N. (2007). Punishment, counter-punishment and sanction enforcement in a social dilemma experiment, *Economic Theory*, 33, pp. 145-167.

Fehr, E. and Gächter, S. (2000). Cooperation and Punishment in Public Goods Experiments, *American Economic Review*, 90(4), pp. 980-994.

Figuières, C., Willinger, M. and Masclet, D. (2009). *Weak moral motivation leads to the decline of voluntary contributions*, Discussion Paper, University of Montpellier I.

Fischbacher, U., Gächter, S. and Fehr, E. (2001). Are people conditionally cooperative? Evidence from a public goods experiment, *Economics Letters*, 71, pp. 397-404.

Fischbacher, U. and Gächter, S. (2010). Social Preferences, Beliefs, and the Dynamics of Free Riding in Public Goods Experiments, *American Economic Review*, 100(1), pp. 541-556.

Fritsch, R. (2001). Pick, Georg, *Neue Deutsche Biographie* 20, pp. 418-419.

Funkenbusch, W. W. (1974). From Euler‘s Formula to Pick‘s Formula Using an Edge Theorem, *The American Mathematical Monthly*, 81(6), pp. 647-648.

Gächter, S. and Thöni, C. (2005). Social Learning and Voluntary Cooperation among Like-Minded People, *Journal of the European Economic Association*, 3, pp. 303-314.

Grünbaum, B. and Shephard, G. C. (1993). Pick’s Theorem, *The American Mathematical Monthly*, 100(2), pp. 150-161.

Hadwiger, H. and Wills, J. M. (1975). Neuere Studien über Gitterpolygone, *Journal für die reine und angewandte Mathematik* , 280, pp. 61-69.

Harbaugh, W. T. and Krause, K. (2000). Children‘s Altruism in Public Good and Dictator Experiments, *Economic Inquiry*, 38(1), pp. 95-109.

Herrmann, B. and Thöni, C. (2009). Measuring conditional cooperation: a replication study in Russia, *Experimental Economics*, 12, pp. 87-92.

Hokamp, S. (2011). *Modeling Income Tax Evasion and the Provision of Public Goods - An Agent-based Simulation*, Mimeo, University of Muenster, Germany.

Houser, D. and Kurzban, R. (2002). Revisiting Kindness and Confusion in Public Goods Experiments, *American Economic Review*, 92(4), pp. 1062-1069.

Isaac, R. M. and Walker, J. M. (1988). Group Size Effects in Public Goods Provision: The Voluntary Contribution Mechanism, *Quarterly Journal of Economics*, 103, pp. 179-199.

Isaac, R. M., Walker, J. M. and Williams, A. W. (1994). Group size and the voluntary provision of public goods, *Journal of Public Economics*, 54, pp. 1-36.

Jackson, M. O. and Wilkie, S. (2005). Endogenous Games and Mechanisms: Payments Among Players, *Review of Economic Studies*, 72, pp. 543-566.

Keser, C. and van Winden, F. (2000). Conditional Cooperation and Voluntary Contributions to Public Goods, *Scandinavian Journal of Economics*, 102(1), pp. 23-39.

Kurzban, R. and Houser, D. (2001). Individual Differences in Cooperation in a Circular Public Goods Game, *European Journal of Personality*, 15, pp. S37-S52.

Ledyard, J. O. (1995). Public Goods: A survey of experimental research. In J.H. Kagel and A. E. Roth (eds.), *The Handbook of Experimental Research* , pp. 111-194. Princeton, N. J.: Princeton University Press.

Leuthold, J. H. (1993). A Free Rider Experiment for the Large Class, *Journal of Economic Education*, 24 (4), pp. 353-363.

Masclet, D., Noussair, C., Tucker, S. and Villeval, M. C. (2003). Monetary and Nonmonetary Punishment in the Voluntary Contributions Mechanism, *American Economic Review*, 93(1), pp. 366-380.

Masclet, D., Willinger, M. and Figuières, C. (2009). *Leadership, Reciprocity and Moral Motivation: Experimental Evidence from a Sequential Public Good Game*, Mimeo, University of Montpellier I and University of Rennes I.

Maurice, J., Rouaix, A. and Willinger, M. (2009). *Income Redistribution and Public Good Provision: an Experiment*, Discussion Paper, University of Montpellier I.

McCorkle, S. and Watts, M. (1996). Free Riding Indexes for Ukrainian Economic Teachers, *Journal of Economic Education*, 27(3), pp. 233-237.

Messer, K. D., Zarghamee, H., Kaiser, H. M. and Schulze, W. D. (2007), New hope for the voluntary contributions mechanism: The effect of context, *Journal of Public Economics*, 91, pp. 1783-1799.

Nikiforakis, N. (2008). Punishment and counter-punishment in public good games: Can we really govern ourselves?, *Journal of Public Economics*, 92, pp.91-112.

Olson, M. (1965). *The Logic of Collective Action: Public Goods and the Theory of Groups*, Cambridge (MA): Harvard University Press.

Pick, G. (1899). Geometrisches zur Zahlenlehre, *Sitzungsberichte Lotos (Prague)*19, pp. 311-319.

Pickhardt, M. (2003). *Studien zur Theorie öffentlicher Güter*, Metropolis: Marburg (Germany).

Pickhardt, M. (2005). Teaching Public Goods Theory With a Classroom Game, *Journal of Economic Education*, 36(2), pp. 145-159.

Pickhardt, M. (2006). Fifty Years after Samuelson's 'The Pure Theory of Public Expenditure': What are we left with?, *Journal of the History of Economic Though*, 28(4), pp. 439-460.

Pickhardt, M. (2010). *A few can do - Ethical behavior and the provision of public goods in an agent-based model*, Discussion Paper, University of Muenster, Germany. [download via RePec.org].

Read, R. C. (1972). *A Mathematical Background for Economists and Social Scientists,* Prentice-Hall Series in Mathematical Economics, Inc., Englewood Cliffs, N. J.

Rege, M. and Telle, K. (2004). The impact of social approval and framing on cooperation in public good situations, *Journal of Public Economics*, 88, pp. 1625-1644.

Samuelson, P. A. (1954). The Pure Theory of Public Expenditure, *Review of Economics and Statistics*, 36, pp. 387-389.

Tan, F. (2008). Punishment in a Linear Public Goods Game with Productivity Heterogeneity, *De Economist*, 156(3), pp. 269-293.

Warr, P. G. (1983). The Private Provision of a Public Good is Independent of the Distribution of Income, *Journal of Economics Letters*, 13, pp. 207-211.

Weimann, J. (1994). Individual behaviour in a free riding experiment, *Journal of Public Economics*, 54(2), pp. 185-200.

Wilson, R. K. and Sell, J. (1997). 'Liar, Liar, ...': Cheap Talk and Reputation in Related Public Goods Settings, *Journal of Conflict Resolution*, 41(5), pp. 695-717.

Zeckhauser, R. J. and Weinstein, M. C. (1974). The Topology of Pareto-Optimal Regions With Public Goods, *Econometrica*, 42(4), pp. 643-666.

Zelmer, J. (2003). Linear Public Goods Experiments: A Meta-Analysis, *Experimental Economics*, 6, pp. 299-310.