Preference Heterogeneity and the Long-Term Evolution of Consumption Shares

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Abstract

We analyze an economy with two heterogeneous Epstein-Zin investors. Our focus is on the long-run evolution of their respective (expected) consumption shares and on the conditions under which heterogeneity prevails over long time horizons such that both investors exhibit a substantial consumption share. A small investor can survive by either saving or speculating her way out of extinction, i.e., by either saving more than her counterpart, or by investing into a riskier portfolio with a higher expected rate of return. The first strategy is followed by the investor with the higher elasticity of intertemporal substitution, while the second is applied by the investor with the lower risk aversion. For the case of i.i.d consumption growth we can analytically decide which of the two investors (or both) survive in the infinite horizon limit. With Epstein-Zin preferences there is a whole region of preference parameter combinations for which both investors can survive, as opposed to CRRA, where this is only a knife-edge case. For the case of long-run risk our simulation analysis shows that the region where both investors retain a non-negligible consumption share is larger than in the i.i.d. case. On the other hand, when the investors are less heterogeneous with respect to risk aversion, this region is smaller, both in the i.i.d. and in the long-run risk case.

Keywords: Epstein-Zin utility, long-run risk, preference heterogeneity, consumpotion shares

JEL: G12

1 Introduction and Motivation

Recently models which combine Epstein-Zin (EZ) type preferences and investor heterogeneity have moved to the center of attention in theoretical asset pricing, see, e.g., the papers by Borovička (2015), Gârleanu and Panageas (2014), and Roche (2011).[1] The use of alternative preferences and the deviation from the homogeneity paradigm represent two major steps in the development of this area of research.

In this paper we analyze a long-run risk model with EZ utility and two heterogeneous investors and investigate whether and, if so, under which conditions both investors can retain a non-negligible (average) consumption share in the long run. We show that introducing a stochastic growth rate of aggregate consumption as a state variable has a significant impact on investor survival. There are two basic economic mechanisms at work here. An investor with a currently small consumption share can either choose relatively aggressive asset positions to earn a high risk premium (speculation channel) or save relatively more than her counterpart (savings channel). Intuitively the former will happen when the small investor's risk aversion is smaller than that of her counterpart, while the latter will be triggered by a higher elasticity of intertemporal substitution.

The main reason for the popularity of EZ preferences compared to classical constant relative risk aversion (CRRA) utility functions is that models with the former can solve a number of asset pricing puzzles in a parsimonious way due to the separability of the level of risk aversion and the elasticity of intertemporal substitution (EIS), which are inverses of each other in the CRRA framework. Furthermore, with EZ preferences there are non-zero premia for state variables, like a stochastic consumption growth rate or stochastic volatility. Overall, this makes it possible to generate much more plausible asset pricing implications with EZ than with CRRA.[2]

[1]An early paper which mainly describes a mathematical approach to the problem of finding an equilibrium in an economy with heterogeneous EZ investors is Dumas, Uppal, and Wang (2000).

[2]See e.g. Bansal and Yaron (2004), Eraker and Shaliastovich (2008), Bansal and Shaliastovich (2011), Drechsler (2013), Zhou and Zhu (2009), Bansal and Shaliastovich (2010), Drechsler and Yaron (2011),

At the same time researchers have become interested in investor heterogeneity, where this term can relate to a number of investor characteristics, most notably beliefs and preference parameters like risk aversion or the subjective rate of time preference. Investor heterogeneity can, similar to alternative preferences like EZ, contribute to a solution of asset pricing puzzles, e.g. by offering at least a partial explanation for the excess volatility of prices relative to fundamentals. Furthermore, it can explain trading between investors, which would not exist in a model with a homogeneous representative investor.[3]

Heterogeneous investors will make different consumption and investment decisions through time. This implies that in general there is a certain probability that some groups of investors will become extinct in the long run, i.e., that their consumption share goes to zero as the time horizon in the model goes to infinity. Especially in the most popular variant of heterogeneous investor models with only two types of agents extinction of one type is obviously a very important issue. If one of the two investors vanishes the model becomes trivial, since the initially heterogeneous economy will then become equivalent to a one investor setup.

This problem already exists in the simplest possible setup with i.i.d. consumption growth and CRRA preferences. For the case of identical beliefs Bhamra and Uppal (2014) show that both investors only survive if a very special condition holds which relates the difference in the two investors' time preference rates to the difference in risk aversion. If this 'knife edge condition' is not satisfied, the consumption share of one type of investors goes to zero, i.e., she faces extinction. To make sure that all implications of heterogeneity carry over to the infinite-horizon case it is thus of importance to identify those parameter sets which guarantee that both investor types remain present in the economy.

In our paper we extend the analysis of economies with heterogeneous EZ investors relative to each of the papers cited above in at least one dimension. Gârleanu and Panageas

Wachter (2013), and Benzoni, Collin-Dufresne, and Goldstein (2011).

[3]See e.g. Dumas (1989), Wang (1996), Dieckmann and Gallmeyer (2005), Longstaff and Wang (2012), Yan (2008), Bhamra and Uppal (2014), Bhamra and Uppal (2009), and Kogan, Ross, Wang, and Westerfield (2011).

(2014) study a model with overlapping generations and i.i.d. consumption growth and provide a detailed analysis of the impact of heterogeneous risk aversion and heterogeneous EIS on asset pricing moments. We consider a more general model with time-varying consumption growth as a long-run risk factor[4] and furthermore allow the individual EIS parameters to take on values above one. While in Gârleanu and Panageas (2014) the two heterogeneous investor types will always be present by construction, we assume infinitely-lived investors, so that long-run survival is no longer guaranteed as in the OLG-setup.

Borovička (2015) investigates the issue of survival in a heterogeneous agent economy in which the two investors differ with respect to their beliefs, but are otherwise identical. He shows that in contrast to the CRRA case also the investor with the more biased beliefs can survive in the long run and interprets this as strong support for the use of preferences where risk aversion is separable from EIS. Our model is more general due to the inclusion of the long-run risk process, and we discuss the matter of investor survival and the long-run evolution of average consumption shares in a setting without differences in beliefs, but differences in preferences. Similar to Borovička (2015) we find that there are parameter *areas* as opposed to the case of 'knife-edge' parameter *combinations* (i.e., lines) where both investors can survive in the long run.

Compared to Roche (2011) we extend the model by relaxing the i.i.d. structure of consumption growth and by enlarging the asset menu the agents can use to trade. By allowing the investors to hold also a locally risk-free bond we are able to separate the decision to save from the decision to take more risk. This separability is key to the main result of our analysis, namely that the small investor can either speculate or save her way out of extinction, so that with a limited asset choice as in Roche (2011) it is impossible that both investors can survive.

With respect to the basic economic mechanisms there are two main types of invest-

[4]The inclusion of a different state variable like stochastic volatility would also be a possible choice to move away from i.i.d. consumption growth. To introduce a stochastic growth rate first seems to us as the natural way to go, following Bansal and Yaron (2004).

ment and portfolio decisions which can help a small investor to retain a non-negligible expected consumption share in the long run. As documented in Borovička (2015), she can either save or speculate 'her way out of extinction'. The former happens when the small investor's wealth-consumption ratio is higher than the large investor's, the latter means that she will tend to hold the more aggressive portfolio earning the higher risk premium.

The main driving force behind a higher wealth-consumption ratio is a higher EIS. On the other hand an investor will hold a portfolio with a higher risk premium when her risk aversion is lower, so that the exposure of wealth to consumption risk and long run growth risk, both of which have a positive market price of risk, will be larger. The most important result is that both investors survive when their respective 'advantage' with respect to one preference parameter is not over-compensated by their 'disadvantage' with respect to the other. When investor 1 is less risk-averse and investor 2 has the higher EIS, then investor 1 can speculate her way out of extinction when she is small (given that her savings rate is not too low, i.e., that her EIS is not too low), while investor 2 can support her survival as the small agent by saving more (given that the expected return on her risky portfolio is not too low, i.e., that her degree of risk aversion is not too high).

An important contribution of our analysis concerns the impact of a state variable on the long-run evolution of average consumption shares. With EZ preferences long-run growth risk is priced and has a positive premium when investors have a preference for early resolution of uncertainty. All else equal, the average consumption share for the less risk-averse investor tends to increase over time, since she now benefits from the higher expected return on her wealth due to the additional premium on growth risk. For the more risk-averse agent the opposite is true, since she has a smaller exposure to growth risk, resulting in a lower expected return on her wealth. To have her average consumption share increase over time, she thus needs a higher EIS.

In the case of a stochastic expected consumption growth there is no longer a closed-form condition for investor survival, so that the evolution of average consumption shares has to be investigated by means of a Monte-Carlo simulation. The results show that the range of scenarios where the less risk-averse agent benefits from the higher expected return on the risky portfolio is very wide. Although the other investor has to compensate this with a higher EIS, the region of EIS-combinations where both investors' average consumption shares increase from initially small values nevertheless becomes larger than in the case without a state variable.

An important implication of our analysis is therefore that in a 'full-fledged' long-run risk model with a number of state variables in addition to a stochastic growth rate of consumption investor heterogeneity can be sustained for a wide variety of preference parameter combinations.

The remainder of this paper is organized as follows. In Section 2, we introduce the model setup. Section 3 contains a formal analysis of the equilibrium. In Section 4 we present numerical results for our model without a state variable and with long-run growth risk. Section 5 concludes.

2 Model Setup

2.1 Endowment

We assume that aggregate endowment C follows the stochastic process

$$\begin{aligned} dC_t &= \mu_C(X_t)C_t dt + \sigma_C' C_t dW_t \\ dX_t &= -\kappa X_t dt + \sigma_X' dW_t, \end{aligned}$$

where $\mu_C(X_t) = \overline{\mu}_C + X_t$. W is a two-dimensional Wiener process, and σ_C and σ_X are volatility vectors in $\mathbb{R}^2$. X is a state variable ('long-run risk') which introduces a stochastic component into the growth rate of aggregate consumption. In the following, we will assume that consumption and the long-run growth rate are locally uncorrelated, i.e., $\sigma_C' \equiv (\sigma_c, 0)$ and $\sigma_X' \equiv (0, \sigma_x)$.

The inclusion of a state variable like X is an important extension of the i.i.d. model when general EZ preferences are considered and creates a 'true' long-run risk model. In contrast to the CRRA case there will in general be a non-zero market price of risk for the innovations in X, which helps to generate a sizeable equity risk premium.

2.2 Investors

There are two heterogeneous investors i ($i = 1, 2$) with recursive utility of the Epstein-Zin (EZ) type.[5] The indirect utility of investor i at time t is

$$J_{i,t} = E_t \left[\int_t^\infty f_i(C_{i,s}, J_{i,s}) ds \right],$$

where $C_{i,t}$ is investor i's consumption at time t, and f_i denotes the aggregator function.

$$f_i(C_i, J_i) = \frac{\beta_i C_i^{1-\frac{1}{\psi_i}}}{\left(1 - \frac{1}{\psi_i}\right) \left[(1-\gamma_i) J_i\right]^{\frac{1}{\theta_i}-1}} - \beta_i \theta_i J_i. \tag{1}$$

In terms of parameters $\gamma_i > 0$ stands for the degree of relative risk aversion, $\psi_i > 0$ is the elasticity of intertemporal substitution (EIS), and $\beta_i > 0$ is the time preference rate. The classical CRRA preferences are obtained as a special case of EZ preferences by setting $\psi_i = 1/\gamma_i$. Concerning risk aversion we assume with only a minor loss of generality that $\gamma_i > 1$. Note that we do not allow the EIS of any investor to be equal to one, i.e., $\psi_i \neq 1$ for $i = 1, 2$. As usual, $\theta_i = \frac{1-\gamma_i}{1-\frac{1}{\psi_i}}$.

We will write $\bar{\gamma}$ for the average risk aversion and $\bar{\psi}$ for the average EIS, where

$$\begin{aligned} \bar{\gamma} &\equiv \frac{1}{\omega \frac{1}{\gamma_1} + (1-\omega)\frac{1}{\gamma_2}} \\ \bar{\psi} &\equiv \omega \psi_1 + (1-\omega)\psi_2, \end{aligned}$$

and where ω is the first investor's consumption share, i.e., $\omega = C_1/(C_1+C_2)$. Analogously, the second investor's share is given by $1 - \omega$.

[5] See Duffie and Epstein (1992) for details on this preference specification in a continuous-time model. Benzoni, Collin-Dufresne, and Goldstein (2011) present an application in the context of option pricing.

Three aspects of the above preference specification are important for our analyses. First, there is the investors' aversion to risk across states, which increases in γ_i. Second, there is the investors' aversion to variation in consumption over time, measured by ψ_i. Here a higher value means more tolerance for, i.e., less aversion to, variation in consumption over time. Third, the investors may have a preference for either early or late resolution of uncertainty. In case the difference $\gamma_i - \psi_i^{-1}$ is positive investor i exhibits a preference for early resolution of uncertainty. The larger the difference the more pronounced this preference.

2.3 Asset Markets

With two heterogeneous investors, market completeness becomes an issue. We assume complete markets by allowing the investors to trade the claim to aggregate consumption, which is in unit net supply (and which we simply call 'the stock'), the money market account (in zero net supply) as well as a contingent claim like an option the value of which is linked to the state variable X (also in zero net supply). The main reasons for choosing to work with complete markets (in addition to computational ease) is that we want to focus on the consequences of investor heterogeneity, not on the implications of market incompleteness. On an incomplete market the investors are restricted to the package of consumption and X-risk represented by the stock, and this restriction has an impact on the equilibrium outcome. It would thus be necessary to analyze a wide range of scenarios in terms of the relative importance of the two risk factors to isolate the implications of heterogeneity from those of market incompleteness. Furthermore, the focus of our investigation is on highly aggregated equity-like assets (the stock market as a whole), so that assuming the existence of an active options market seems natural.

3 Survival and Long-Run Evolution of Consumption Shares

3.1 Structure of the Analysis

Intuitively survival is linked to the behavior of an investor's consumption share when she is small. Roughly speaking the issue here is whether an already small consumption share will on average decrease further or whether there is a chance for the investor to recover. Consequently the following sections contain an analysis of the relevant properties of the small investor's consumption share, i.e., its drift and its volatility. These two quantities are integral parts of the equilibrium solution of our model (together with the individual wealth-consumption ratios and the aggregate pricing kernel), so this is the first main result we are going to derive.

For the case without a state variable the technical conditions for long-run survival can even be stated in closed-form. In terms of the exact computations we rely on the analysis in Borovička (2015) (based on Karlin and Taylor (1981)), adapted to our setting. The purely mathematical representation of the survival condition is, however, only one part of the analysis, since we first and foremost want to gain a deeper understanding of the *economic* mechanisms behind survival and extinction. So we devote a separate section to the economic interpretation of the mathematical results, where we trace the survival condition back to the investment and savings decisions of the small (as compared to the large) investor. In case there is a state variable, such a closed-form survival condition no longer exists, so that we have to resort to Monte-Carlo simulations of the average consumption shares.

3.2 Solving for the Equilibrium

3.2.1 Consumption Sharing Rule

With two heterogeneous investors the equilibrium consumption sharing rule is one of the key outputs of the model. Investor 1's consumption share $\omega = \frac{C_1}{C_1+C_2}$ introduced in Section 2 is our key state variable and exhibits dynamics

$$d\omega_t \quad = \quad \mu_\omega(\omega_t, X_t)dt + \sigma_\omega(\omega_t, X_t)' dW_t.$$

The coefficient functions in this stochastic differential equation have to be determined as part of the solution. Below we will use a more compact notation by setting $\mu_\omega \equiv \mu_\omega(\omega_t, X_t)$ and $\sigma_\omega \equiv \sigma_\omega(\omega_t, X_t)$.

Given the dynamics of aggregate consumption and investor 1's consumption share the evolution of investor 1's individual consumption *level* $C_1 \equiv \omega C$ is obtained via Ito's lemma as

$$\frac{dC_{1,t}}{C_{1,t}} = \mu_{C_1} dt + \sigma'_{C_1} dW_t$$

with drift and volatility coefficients

$$\mu_{C_1} = \mu_C + \frac{1}{\omega}\,\mu_\omega + \frac{1}{\omega}\,\sigma'_\omega \sigma_C \tag{2}$$

and

$$\sigma_{C_1} = \sigma_C + \frac{1}{\omega}\,\sigma_\omega. \tag{3}$$

Analogous expressions are obtained for investor 2's consumption.

3.2.2 Individual Wealth-Consumption Ratios

Given prices each investor maximizes her indirect utility J_i. In the optimum it holds that

$$E_t\left[dJ_{i,t} + f_i(C_{i,t}, J_{i,t})dt\right] \quad = \quad 0, \tag{4}$$

and we set

$$J_{i,t} = \frac{C_{i,t}^{1-\gamma_i}}{1-\gamma_i} \beta_i^{\theta_i} e^{\theta_i v_{i,t}}. \tag{5}$$

Given this specification it can be shown that $v_{i,t} \equiv v_i(\omega_t, X_t)$ is investor i's log wealth-consumption ratio at time t (see Benzoni, Collin-Dufresne, and Goldstein (2011)). Its dynamics follow from Ito's lemma and are given by the stochastic differential equation

$$dv_{i,t} \;=\; \mu_{v_i}(\omega_t, X_t)dt + \sigma_{v_i}(\omega_t, X_t)'dW_t.$$

The drift μ_{v_i} and the volatility σ_{v_i} obviously depend on the partial derivatives of v_i with respect to ω and X, which are given in Appendices A.2.2 and A.2.3. Again, the dependence of μ_{v_i} and σ_{v_i} on ω and X will be suppressed from here on to make the notation more readable.

Plugging the dynamics of the indirect utility function derived from Equation (5) into Equation (4) and simplifying we obtain partial differential equations for the log wealth-consumption ratios v_i $(i = 1, 2)$:

$$0 = e^{-v_i} - \beta_i + \left(1 - \frac{1}{\psi_i}\right)\left(\mu_{C_i} - \frac{1}{2}\gamma_i \sigma'_{C_i}\sigma_{C_i}\right) + \mu_{v_i} + \frac{1}{2}\theta_i \sigma'_{v_i}\sigma_{v_i} + (1-\gamma_i)\sigma'_{C_i}\sigma_{v_i}. \quad (6)$$

3.2.3 Pricing Kernel

As shown in Benzoni, Collin-Dufresne, and Goldstein (2011) investor i's pricing kernel at time t (denoted by $\xi_{i,t}$) is given by

$$\xi_{i,t} \;=\; \beta_i^{\theta_i} e^{-\beta_i\theta_i t - (1-\theta_i)\int_0^t e^{-v_{i,s}}ds} e^{(\theta_i - 1)v_{i,t}} C_{i,t}^{-\gamma_i}. \quad (7)$$

Its dynamics follow directly by an application of Ito's lemma. Simplifying and using Equation (6) for the individual log wealth-consumption ratio gives

$$\begin{aligned} \frac{d\xi_{i,t}}{\xi_{i,t}} \;=\; & -\left[\beta_i + \frac{1}{\psi_i}\mu_{C_i} - \frac{1}{2}\gamma_i\left(1 + \frac{1}{\psi_i}\right)\sigma'_{C_i}\sigma_{C_i} - \frac{1}{2}(1-\theta_i)\sigma'_{v_i}\sigma_{v_i} - (1-\theta_i)\sigma'_{C_i}\sigma_{v_i}\right]dt \\ & - \left[\gamma_i\sigma_{C_i} + (1-\theta_i)\sigma_{v_i}\right]' dW_t \end{aligned} \quad (8)$$

for $i = 1, 2$. From this we obtain investor i's subjective risk-free rate $r_{i,t}$ as

$$r_{i,t} = \beta_i + \frac{1}{\psi_i}\mu_{C_i} - \frac{1}{2}\gamma_i\left(1 + \frac{1}{\psi_i}\right)\sigma'_{C_i}\sigma_{C_i} - \frac{1}{2}(1-\theta_i)\sigma'_{v_i}\sigma_{v_i} - (1-\theta_i)\sigma'_{C_i}\sigma_{v_i}$$

and the (subjective) market prices of risk $\lambda_{i,t}$ as

$$\lambda_{i,t} = \gamma_i\sigma_{C_i} + (1-\theta_i)\sigma_{v_i}.$$

3.2.4 Putting the Pieces Together

We are now in a position to solve for the equilibrium quantities of interest. The investors have to agree on the risk-free rate and the market prices of risk, i.e., on the drift and volatility of the pricing kernel. These functions depend on the individual (log) wealth-consumption ratios, which in turn have to be chosen such that the individual indirect utilities satisfy Equation (4).

In detail we have to solve for the following five quantities, which are all functions of the two state variables ω and X: the investors' individual log wealth-consumption ratios v_1 and v_2 (which have to be chosen such that Equation (6) is satisfied for $i = 1, 2$), the drift function μ_ω of investor 1's consumption share (which follows from equating the subjective risk-free rates), and the two volatilities collected in the vector σ_ω (which follow from equating the subjective market prices of risk for the two Wiener processes driving C and X).

One issue which has to be resolved concerns the boundary conditions for the partial differential equations. For the consumption share ω the domain is the closed interval from zero to one. At the boundaries of this interval the problem reduces to finding the equilibrium in an economy with one investor only (see the appendix for details), so that the wealth-consumption ratio of the large investor is given by the solution in a homogenous economy populated exclusively by this type. For the state variable X there are no 'natural' boundaries, so we have to consider the solution over a sufficiently large interval.

The boundary condition for $\omega \to 0$ assumes no price impact on the part of the small investor. In his proof of this claim Borovička (2015) argues that the drift of $\ln \omega$ is bounded so that a substantially positive consumption share for the small investor can be moved arbitrarily far into the future, which in turn eliminates a potential current price impact via the force of discounting.

3.3 Dynamics of Individual Consumption Shares

3.3.1 Drift and Volatility of Consumption Shares

The vector of volatilities of the consumption share follows from equating the investors' subjective market prices of risk:

$$\gamma_1 \left(\sigma_C + \frac{1}{\omega}\, \sigma_\omega \right) + (1-\theta_1)\sigma_{v_1} \;=\; \gamma_2 \left(\sigma_C - \frac{1}{1-\omega}\, \sigma_\omega \right) + (1-\theta_2)\sigma_{v_2}.$$

Solving for σ_ω, the volatility vector of investor 1's consumption share, yields

$$\sigma_\omega \;=\; \frac{\omega(1-\omega)\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\sigma_C + \frac{\omega(1-\omega)}{\gamma_1\gamma_2}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X}{\frac{1}{\bar{\gamma}} + \frac{\omega(1-\omega)}{\gamma_1\gamma_2}\left[(1-\theta_1)\frac{\partial v_1}{\partial \omega} - (1-\theta_2)\frac{\partial v_2}{\partial \omega}\right]} \tag{9}$$

Consider first the limiting cases $\omega \to 0$ and $\omega \to 1$. At the boundaries for ω the volatility of investor 1's consumption share goes to zero. This is also in line with intuition, since a non-zero volatility would imply that the consumption share can become negative or greater than one.

It is also important to consider the limiting behavior of the normalized consumption share volatility $\frac{1}{\omega}\,\sigma_\omega$. With $\sigma_{C_1} = \sigma_C + \frac{1}{\omega}\sigma_\omega$ this term captures the deviation of individual consumption volatility from the volatility of the aggregate and thus describes risk sharing between investors. The normalized consumption share volatility goes to zero for $\omega \to 1$, but converges to some finite value for $\omega \to 0$ (see Appendix A.2.1). These properties of $\frac{1}{\omega}\,\sigma_\omega$ in the limits are perfectly in line with intuition. The large investor obviously cannot share risk any more, but has to consume C, implying $\sigma_{C_1} = \sigma_C$. The small investor has the best chance for risk-sharing. Due to the fact that her overall share in the economy is so small, her positions barely matter to the large investor.

The drift μ_ω of investor 1's consumption share follows from equating the investors' subjective risk-free rates, which we get from the drift of the pricing kernel in Equation (8). This yields the condition

$$\begin{aligned}
& \beta_1 + \frac{1}{\psi_1}\,\mu_{C_1} - 0.5\gamma_1\left(1 + \frac{1}{\psi_1}\right)\sigma_{C_1}'\sigma_{C_1} - 0.5(1-\theta_1)\sigma_{v_1}'\sigma_{v_1} - (1-\theta_1)\sigma_{C_1}'\sigma_{v_1} \\
= \;& \beta_2 + \frac{1}{\psi_2}\,\mu_{C_2} - 0.5\gamma_2\left(1 + \frac{1}{\psi_2}\right)\sigma_{C_2}'\sigma_{C_2} - 0.5(1-\theta_2)\sigma_{v_2}'\sigma_{v_2} - (1-\theta_2)\sigma_{C_2}'\sigma_{v_2}.
\end{aligned}$$

Plugging in μ_{C_i} from Equation (2) and solving for μ_ω yields

$$\begin{aligned}\mu_\omega &= -\sigma'_\omega\sigma_C + \frac{\omega(1-\omega)\psi_1\psi_2}{\bar{\psi}} \\ &\quad\times\Bigg\{\beta_2-\beta_1+\left(\frac{1}{\psi_2}-\frac{1}{\psi_1}\right)\mu_C \\ &\quad - 0.5\gamma_2\left(1+\frac{1}{\psi_2}\right)\sigma'_{C_2}\sigma_{C_2} - 0.5(1-\theta_2)\sigma'_{v_2}\sigma_{v_2} - (1-\theta_2)\sigma'_{C_2}\sigma_{v_2} \\ &\quad + 0.5\gamma_1\left(1+\frac{1}{\psi_1}\right)\sigma'_{C_1}\sigma_{C_1} + 0.5(1-\theta_1)\sigma'_{v_1}\sigma_{v_1} + (1-\theta_1)\sigma'_{C_1}\sigma_{v_1}\Bigg\}. \qquad (10)\end{aligned}$$

In what follows we will assume that the economy has a positive average growth rate. More precisely, we assume $\overline{\mu}_C > 0$ and a long-run mean of zero for the stochastic growth rate X. Furthermore, we restrict the analysis to the case when both investors' wealth-consumption ratios are finite (see the appendix for details). As pointed out by Borovička (2015), this implies a lower bound on β_i.

3.3.2 Technical Conditions for Survival

For the reasons stated in the introduction we are mostly interested to find out for what parameter and preference scenarios both investors survive in the long run, since for these cases we have a well-defined model with heterogeneous agents also for an infinite horizon, i.e., in the steady state. In our interpretation both investors survive, if in the limit as $t \to \infty$, the probabilities that ω goes to zero or to one are both zero, i.e., it must hold that $P(\lim_{t\to\infty}\omega_t = 0) = P(\lim_{t\to\infty}\omega_t = 1) = 0$. In the model without a state variable, we can rely on the results of Karlin and Taylor (1981). They show that the probability that the stochastic process for ω reaches zero in the limit can be translated into a condition for the drift and the volatility of ω or, equivalently, of $\ln\omega$. Intuitively, investor 1 will survive if the drift of $\ln\omega$ is positive in the neighborhood of $\omega = 0$. Analogously for investor 2 the drift of $\ln(1-\omega)$ must be positive near $\omega = 1$. The exact condition for investor 1 to escape extinction is thus

$$\lim_{\omega\to 0}\left[\frac{\mu_\omega(\omega)}{\omega} - \frac{1}{2}\frac{\sigma^2_\omega(\omega)}{\omega^2}\right] > 0. \qquad (11)$$

One can show (see Appendix A.4) that this is equivalent to the condition

$$\begin{aligned}
&\beta_1 + \frac{1}{\psi_1}\left(\mu_C - 0.5\sigma_C'\sigma_C\right) + 0.5\left(\gamma_1 - \frac{1}{\psi_1}\right)(\gamma_1 - 1)\left(\frac{\gamma_2}{\gamma_1}\right)^2 \sigma_C'\sigma_C \\
\leq\ & \beta_2 + \frac{1}{\psi_2}\left(\mu_C - 0.5\sigma_C'\sigma_C\right) + 0.5\left(\gamma_2 - \frac{1}{\psi_2}\right)(\gamma_2 - 1)\,\sigma_C'\sigma_C. \qquad (12)
\end{aligned}$$

The condition for investor 2's survival is completely analogous, one merely has to switch the indices '1' and '2' in (12). This implies that the analogous condition for investor 2 is *not* obtained by simply reverting the inequality sign in (12), and the reason for this is that the third terms on the two sides of the inequality are not mirror images of each other.

For the model with long-run growth risk, the analysis becomes somewhat more involved. The drift of $\ln\omega$ now depends on the state variable X. Intuitively, it then matters whether the investor can escape extinction *on average* when she becomes small. This implies that one needs to know the joint distribution of ω and X. To get the intuition, consider the case where the drift of $\ln\omega$ depends on X (which is true in our model). If ω goes to zero only for those values of X which come with a negative drift, then investor 1's consumption share will decrease over time. If on the other hand very small consumption shares for investor 1 tend to occur for those values of X which imply a positive drift for ω, investor 1 will see her consumption share increase again when she is small.[6]

3.4 Economic Analysis of Survival

The fundamental economic quantities of interest are the individuals' consumption and investment decisions. This is why we will now analyze the drift of $\ln\omega$ from an economic point of view. Even if there is a closed-form condition for survival only in the case of i.i.d. consumption growth, the dynamics of $\ln\omega$ will obviously also matter for the long-run evolution of consumption shares when expected consumption growth is stochastic.

The survival condition (11) is a statement about the drift and volatility of the

[6]We will show below that the drift of the log consumption share and the state variable X indeed exhibit non-zero correlation. See Tables 9 and 10.

consumption share ω. The dynamics of individual consumption in Equations (2) and (3) imply

$$\frac{\mu_\omega(\omega,x)}{\omega}-\frac{1}{2}\frac{\sigma_\omega^2(\omega,x)}{\omega^2}=\left(\mu_{C_1}-\frac{1}{2}\sigma_{C_1}'\sigma_{C_1}\right)-\left(\mu_C-\frac{1}{2}\sigma_C'\sigma_C\right). \tag{13}$$

In the limit as $\omega\to 0$ the economy basically only consists of the large investor 2, so that $C=C_2$, and the above equation becomes

$$\lim_{\omega\to 0}\left(\frac{\mu_\omega(\omega,x)}{\omega}-\frac{1}{2}\frac{\sigma_\omega^2(\omega,x)}{\omega^2}\right)=\left(\mu_{C_1}-\frac{1}{2}\sigma_{C_1}'\sigma_{C_1}\right)-\left(\mu_{C_2}-\frac{1}{2}\sigma_{C_2}'\sigma_{C_2}\right).$$

This means that, for investor 1 to have a chance to survive, the average drift of her individual log consumption must be greater than that of her counterpart.

How is survival related to the dynamics of individual wealth? With $V_i\equiv C_ie^{v_i}$ $(i=1,2)$ denoting investor i's wealth, its drift and volatility follow from an application of Ito's lemma as

$$\begin{aligned}\mu_{V_i}&=\mu_{C_i}+\mu_{v_i}+\frac{1}{2}\sigma_{v_i}'\sigma_{v_i}+\sigma_{v_i}'\sigma_{C_i}\\ \sigma_{V_i}&=\sigma_{C_i}+\sigma_{v_i},\end{aligned}$$

and the same is true for aggregate wealth $V\equiv Ce^v$. Using these equations together with (13) yields

$$\frac{\mu_\omega(\omega,x)}{\omega}-\frac{1}{2}\frac{\sigma_\omega^2(\omega,x)}{\omega^2}=\left(\mu_{V_1}-\frac{1}{2}\sigma_{V_1}'\sigma_{V_1}-\mu_{v_1}\right)-\left(\mu_V-\frac{1}{2}\sigma_V'\sigma_V-\mu_v\right).$$

Given that $\mu_{V_i}+e^{-v_i}=r+\sigma_{V_i}'\lambda$, i.e., the expected return on individual wealth plus the current individual consumption-wealth ratio equals the risk-free rate plus the risk premium, we can further deduce that

$$\frac{\mu_\omega(\omega,x)}{\omega}-\frac{1}{2}\frac{\sigma_\omega^2(\omega,x)}{\omega^2}=\left(\sigma_{V_1}'\lambda-e^{-v_1}-\mu_{v_1}-\frac{1}{2}\sigma_{V_1}'\sigma_{V_1}\right)-\left(\sigma_V'\lambda-e^{-v}-\mu_v-\frac{1}{2}\sigma_V'\sigma_V\right).$$

For $\omega\to 0$ we can replace aggregate quantities in the second term in parentheses by those for investor 2, so that we ultimately obtain

$$\begin{aligned}&\lim_{\omega\to 0}\left(\frac{\mu_\omega(\omega,x)}{\omega}-\frac{1}{2}\frac{\sigma_\omega^2(\omega,x)}{\omega^2}\right)\\ &=\left(\sigma_{V_1}'\lambda-e^{-v_1}-\mu_{v_1}-\frac{1}{2}\sigma_{V_1}'\sigma_{V_1}\right)-\left(\sigma_{V_2}'\lambda-e^{-v_2}-\mu_{v_2}-\frac{1}{2}\sigma_{V_2}'\sigma_{V_2}\right).\end{aligned} \tag{14}$$

Decomposing the right-hand side of Equation (14) gives us the opportunity to identify the fundamental economic mechanisms behind survival.

First, in the spirit of Borovička (2015), we look at the difference $\sigma_{V_1}'\lambda - \sigma_{V_2}'\lambda$, the difference in risk premia earned on individual wealth. It contributes to investor 1's chances to survive if she is the one holding the portfolio with the higher risk premium. She then tries to 'speculate her way out of extinction'. This constitutes what Borovička (2015) calls the 'risk premium channel'.

We assume a preference for early resolution of uncertainty for both investors, i.e., $\gamma_i > \psi_i^{-1}$ for $i = 1, 2$. The market prices of risk are then positive for both consumption risk and long-run growth risk, so that the expected return is the higher the higher the exposures of current wealth to these two risk factors.

When ω goes to 0, investor 1 is small and, accordingly, investor 2 is large. In Appendices A.2.1 and A.2.2 we compute the limiting volatilities of consumption and of the log wealth-consumption ratio. Putting the results together we obtain

$$\lim_{\omega\to 0}\sigma_{V_1} = \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{\partial v_1}{\partial x}\sigma_X + \frac{1}{\gamma_1}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X \tag{15}$$

$$\lim_{\omega\to 0}\sigma_{V_2} = \sigma_C + \frac{\partial v_2}{\partial x}\sigma_X, \tag{16}$$

where the partial derivatives with respect to x are evaluated in the limit as $\omega \to 0$.

We can now analyze the exposures to the risk factors in the model one by one. It immediately follows that investor 1 has a higher exposure to consumption risk and thus earns the higher premium on this factor if she is less risk averse, i.e., if $\gamma_1 < \gamma_2$.

For X-risk the picture is not so clear. We start our analysis by looking at the last term on the right-hand side of (15), which is not present in (16). This term corresponds to the exposure of investor 1's consumption to X-risk. It is obvious that the corresponding exposure for investor 2 (who is large) must be zero, since her consumption basically equals aggregate consumption, and so do the associated risks. If we want to know whether investor 1's consumption exposure to X is higher than investor 2's, the answer depends on whether the term in square brackets is positive or negative. It is impossible to determine its sign analytically, but we can rely on an approximation described in Appendix A.3.2, yielding

$$\begin{aligned}\lim_{\omega\to 0}\left[(1-\theta_2)\frac{\partial v_2}{\partial x}-(1-\theta_1)\frac{\partial v_1}{\partial x}\right] &\approx \frac{\gamma_2-\frac{1}{\psi_2}}{e^{-v_2}+\kappa_x}-\frac{\gamma_1-\frac{1}{\psi_1}}{e^{-v_1}+\kappa_x}\frac{\psi_1}{\psi_2}\\ &\approx \frac{1}{\psi_2}\cdot\frac{\gamma_2\psi_2-\gamma_1\psi_1}{e^{-\bar{v}}+\kappa_x},\end{aligned}$$

where $\bar{v}$ is the log wealth-consumption ratio assumed to be common to the two investors in the approximation, i.e., $\bar{v}\approx v_1(0)\approx v_2(0)$.

Based on this approximation the exposure of the small investor's consumption to X is positive, if and only if $\gamma_1\psi_1<\gamma_2\psi_2$. This will be true, e.g., when investor 1 has CRRA preferences (implying that $\gamma_1\psi_1=1$) and investor 2 has a preference for early resolution of uncertainty, implying $\gamma_2>\psi_2^{-1}$ and thus $\gamma_2\psi_2>1$. To get the intuition, note that the premium for X-risk is set by the large investor 2 and increases in her preference for early resolution of uncertainty. The small investor 1 then takes a long position in X-risk if that premium is attractive to her, i.e., if she has a less pronounced preference for early resolution of uncertainty.

It now remains to look at the respective second terms on the right-hand sides of Equations (15) and (16). They represent the response of the individual wealth-consumption ratios to changes in X. Appendix A.3.1 provides approximate expressions for the partial derivatives of the log wealth-consumption ratios:

$$\begin{aligned}\lim_{\omega\to 0}\frac{\partial v_1}{\partial x} &\approx \frac{1-\frac{1}{\psi_1}}{e^{-v_1}+\kappa_x}\cdot\frac{\psi_1}{\psi_2}\\ \lim_{\omega\to 0}\frac{\partial v_2}{\partial x} &\approx \frac{1-\frac{1}{\psi_2}}{e^{-v_2}+\kappa_x}.\end{aligned}$$

For $v_1(0)\approx v_2(0)\approx\bar{v}$ investor 1's exposure will be greater than investor 2's if $\psi_1>\psi_2$.

Collecting the above results we are now in a position to make a statement about the difference between the two investors in terms of the risk premia earned on their respective individual wealth. We find that

$$\lim_{\omega\to 0}(\sigma_{V_1}-\sigma_{V_2})'\lambda \approx \left[\left(\frac{\gamma_2}{\gamma_1}-1\right)\sigma_C + \frac{\frac{\gamma_2}{\gamma_1}-1}{e^{-\bar{v}}+\kappa}\sigma_X\right]'\lambda, \tag{17}$$

which is positive for $\gamma_1 < \gamma_2$. So it is ultimately the less risk-averse investor who earns the higher risk premium on her wealth when she is small. Since a higher risk premium is associated with more aggressive investment behavior, this effect can be described as 'speculating one's way out of extinction.'

Analogously to Borovička (2015) there is also a phenomenon like 'saving one's way out of extinction'. To see how the mechanism works here, consider again Equation (14), this time the term $(e^{-v_2}-e^{-v_1})+(\mu_{v_2}-\mu_{v_1})$. If this expression is positive then investor 1 either consumes less out of her wealth than investor 2 (her consumption-to-wealth ratio e^{-v_1} is smaller than that of investor 2) or postpones consumption more than investor 2 (hier wealth-consumption ratio grows more slowly on average, i.e., she will consume more out of her wealth than investor 2 in the future, but not now), or both. Both things imply that investor 1 is more willing to save than investor 2, and according to Equation (14) this improves her chances to survive by saving her way out of extinction.

We now link the investor 1's higher willingness to save to differences in preferences between investors. Substituting for e^{-v_i} $(i=1,2)$ using Equation (6) we can see that

$$\begin{aligned}
&(e^{-v_2}-e^{-v_1})+(\mu_{v_2}-\mu_{v_1})\\
&= \beta_2-\beta_1\\
&\quad+\left(1-\frac{1}{\psi_1}\right)\left(\mu_{C_1}-\frac{1}{2}\gamma_1\sigma_{C_1}'\sigma_{C_1}\right)-\left(1-\frac{1}{\psi_2}\right)\left(\mu_{C_2}-\frac{1}{2}\gamma_2\sigma_{C_2}'\sigma_{C_2}\right)\\
&\quad+\frac{1}{2}\theta_1\sigma_{v_1}'\sigma_{v_1}-\frac{1}{2}\theta_2\sigma_{v_2}'\sigma_{v_2}\\
&\quad+(1-\gamma_1)\sigma_{C_1}'\sigma_{v_1}-(1-\gamma_2)\sigma_{C_2}'\sigma_{v_2}.
\end{aligned}$$

Looking at the first two lines on the right-hand side we see that investor 1's propensity to save and thus her prospects of survival first of all improve when she has the lower subjective rate of time preference, i.e., when she is more patient and thus more willing to postpone consumption. Second, ignoring differences in the drifts and volatilities of personal consumption, investor 1 also benefits from a higher EIS, i.e., when she is ready to accept a 'less flat' consumption path. In a growing economy she will then be ready to have a higher growth rate in her personal consumption, which implies lower consumption and thus higher savings today. The remaining two lines are related to the conditional variances of the individual wealth-consumption ratios and to the covariances of the v_i with individual consumption.

Given these two fundamental economic mechanisms behind survival we now analyze for which preference combinations both investors can survive. If one of the investors has a 'double advantage', i.e., exhibits both a lower risk aversion and a higher EIS than her counterpart, then this investor will dominate in the long run. If on the other hand each investor has an advantage in one of the two preference dimensions, they may both survive in the long run.

Joint survival of both investors is thus possible, if the investor with the lower risk aversion (who will try to speculate her way out of extinction) also has the lower EIS (so that it is the other investor, who will try to save her way out of extinction). For joint survival to actually occur, however, the two advantages must be somehow balanced. When the less risk-averse investor is close to extinction she will hold the riskier portfolio, but she will also save less due to her lower EIS. In case she saves too little, however, she will nevertheless not survive. An exactly analogous logic holds for the opposite case when the small investor has the higher EIS and thus saves more, but may invest into too conservative a portfolio.

This possibility that either investor can have an advantage with respect to one of the two preference characteristics is the key difference to the CRRA case. Since there γ has to equal $1/\psi$, having the lower risk aversion necessarily means to have the higher EIS, so that there are no offsetting effects of one preference parameter with respect to the other. This is turn means that survival becomes the 'knife-edge' result mentioned above, which is only possible if two symmetric inequalities are satisfied at the same time, implying an equality.[7]

This becomes directly obvious when we look at the survival condition (12) in the case without a state variable. As noted above the two sides of the inequality are not exactly symmetric, but in the CRRA case with $\gamma_i = \psi_i^{-1}$ for $i = 1, 2$ the third term causing the initial asymmetry vanishes, so that the condition for investor 1's survival now reads $\beta_1 + \gamma_1 (\mu_C - 0.5\sigma_C' \sigma_C) \leq \beta_2 + \gamma_2 (\mu_C - 0.5\sigma_C' \sigma_C)$. For investor 2, swapping the indices is now equivalent to reverting the inequality sign, so that the only situation in which both investors can survive is when $\beta_1 + \gamma_1 (\mu_C - 0.5\sigma_C' \sigma_C) = \beta_2 + \gamma_2 (\mu_C - 0.5\sigma_C' \sigma_C)$, the knife-edge case mentioned above.

4 Quantitative Analysis

4.1 I.i.d. Consumption Growth

The setup without a state variable mainly serves as a benchmark against which we will later compare the results for the case with long-run risk. The restricted model is of course easier to solve, and it substantially helps to develop intuition about the economic effects in a heterogeneous agent economy. For all the numerical analyses below we assume $\overline{\mu}_C = 0.02$, $\sigma_C = 0.0252$, $\beta_1 = \beta_2 = 0.1$, $\gamma_1 = 4$, and $\gamma_2 = 10$. The analysis below focuses on the implications of different combinations of the two EIS parameters, ψ_1 and ψ_2, which we vary between 0.4 and 1.6 each. To get a feel for the impact of heterogeneity with repsct to risk aversion, we compare the above base case to a scenario with $\gamma_1 = 6$ and $\gamma_2 = 8$.

The weak inequality condition for investor 1's survival in the model without long-run growth risk is shown above in (12), which states a relation between all the preference-related parameters of the model and the consumption dynamics. Figure 1 focusses on the (ψ_1, ψ_2)-combinations and shows the regions where investor 1, investor 2, or both survive.[8]

[7] Of course, the individual rates of time preference are an additional degree of freedom in the preference specification of the two investors, but we only focus on risk aversion and the EIS.

[8] In case the small investor's wealth-consumption ratio turned out to be infinite we proceed in the

In terms of investor survival the graphs show that it is of first order importance in this setup to have the higher EIS. We see that the less risk averse investor 1 'can afford' to have a lower EIS than investor 2 and still survive (as long as her EIS is not too low), indicated by the shaded area above the 45 degree line in the left graph in Figure 1. Investor 2 on the other hand needs a somewhat larger EIS to survive than her less risk-averse counterpart, as shown in the middle graph in Figure 1.

We now want to trace the analytical survival condition back to the underlying economics, i.e., back to the consumption and savings decisions by the small investor. Investor 1 is less risk-averse and will therefore try to speculate her way out of extinction, when she is small. Looking at the risk exposure of the investors' portfolios (and assuming a bounded wealth-consumption ratio for the small investor) we see that the volatility of the small investor 1's individual consumption (and thus also the consumption risk exposure of her wealth in the case without a state variable) is given by $\sigma_{C_1}(0) = \frac{\gamma_2}{\gamma_1}\sigma_C$, as compared to σ_C for the large investor 2 (see Appendix A.2.1). The small investor thus, unsurprisingly, takes on additional consumption risk since she is less risk averse, i.e., since $\gamma_1 < \gamma_2$. In this case the market price of consumption risk $\gamma_2\sigma_C$ set by the large investor is attractive to her and induces her to take additional risk. She thus earns a higher expected return on her wealth than the large, more risk-averse investor.

By the same logic the small investor 2 will hold a less risky portfolio than her large counterpart, which can also be seen from a comparison of the two investors' consumption volatilities, since $\sigma_{C_2}(1) = \frac{\gamma_1}{\gamma_2}\sigma_C < \sigma_C$ and $\sigma_{C_1}(1) = \sigma_C$.

The second element of long-run survival is the investors' savings behavior. To see whether the small investor can save her way out of extinction we therefore need to compare the investors' wealth-consumption ratios, as discussed above. Solving the partial differ-

analysis of investor 1's survival as follows: Assume Condition (12) has indicated survival for a given combination of ψ_1 and ψ_2. Then we also label any other combination with the same ψ_2 and a higher ψ_1 as 'survival'. Given the economic mechanisms described in Section 3 this seems a sensible approach. In the analysis of investor 2's survival we proceeded in a completely analogous fashion. This approach is feasible, whenever the left-hand side of (12) is decreasing in ψ_1. This is true for our parametrization.

ential equation in Appendix A.2.4 for the wealth-consumption ratio $v_2(0)$ of the large investor 2 in the case without a state variable yields

$$e^{v_2(0)} = \frac{1}{\beta_2 - \left(1 - \frac{1}{\psi_2}\right)\left(\mu_C - \frac{1}{2}\gamma_2\sigma_C^2\right)}.$$

Not surprisingly, this is also the solution in a homogeneous economy with only this investor type. Given our basic parameters $v_2(0)$ (and also $v_1(1)$) is indeed bounded for all the combinations of preference parameters we consider.

The wealth-consumption ratio of the small investor 1 is (see Appendix A.2.4)

$$e^{v_1(0)} = \frac{1}{\psi_1\beta_1 + (1-\psi_1)\beta_2 - \left(1 - \frac{1}{\psi_1}\right)\frac{\psi_1}{\psi_2}\left(\mu_C - 0.5\gamma_2\sigma_C^2\right) - \frac{1}{2}\left(\psi_1 - 1\right)\gamma_2\left(\frac{\gamma_2}{\gamma_1} - 1\right)\sigma_C^2}, \quad (18)$$

given that the denominator is positive. It will be negative when investor 1 is much more patient, or exhibits a much lower risk aversion, or has a much higher EIS. In these cases the investment opportunities set by the large investor 2 are overly attractive to the small investor, and her wealth-consumption ratio becomes infinite.[9] Again the analysis for the case $\omega \to 1$ is completely analogous.

The shaded regions in Figure 2 represent the combinations of ψ_1 and ψ_2 for which the wealth-consumption ratio of the small investor is indeed bounded. We see that an unbounded wealth-consumption ratio for the small investor occurs when her EIS is greater than one and also sufficiently larger than the EIS of her counterpart. Note that the region of bounded wealth-consumption ratios is slightly smaller for the less risk-averse investor (here investor 1).

In terms of comparing the two individual wealth-consumption ratios the shaded regions in Figure 3 show the combinations of parameters for which the small investor's wealth-consumption ratio is greater than the large investor's, i.e., they indicate the cases when the smaller investor indeed saves more than her counterpart. In the left graph the less risk-averse investor 1 is small, in the right picture it is investor 2. One can immediately see that differences in the investors' EIS explain almost everything here. Basically whenever

[9]Note that her *wealth* still remains finite, though.

the small investor has the higher EIS she also has the higher wealth-consumption ratio and thus saves more. The (albeit small) differences between the two graphs in Figure 3 are caused by differences in risk aversion, with the shaded region being slightly larger for the less risk averse investor. Compared to the impact of the EIS, however, these effects of differences in risk aversion appear rather small.

Comparing Figure 3 to Figure 1 highlights the relation between survival and savings behavior. For the less risk-averse investor, a comparison of the left graphs in the two figures shows that the area where the small investor saves more is a strict subset of the region where she can survive. The former is the region where the investor both saves and speculates her way out of extinction. The difference between the two shaded areas thus represents the cases where the investor only speculates her way out of extinction by taking a larger position in consumption risk. Here the disadvantage of saving less can still be offset by the higher risk premium earned on wealth. Analogously when the small investor is more risk averse the area for which she can survive is slightly smaller than the region where she has the higher savings. To offset her disadvantage from earning a lower expected rate of return on her wealth, her savings rate has to be larger than the one of the other investor by some finite amount.

Most importantly we find a significant overlap of the shaded areas in the left and middle graph in Figure 1 (shown separately in the right graph). This implies that there are indeed combinations of ψ_1 and ψ_2 for which both investors will survive. A similar result has been shown by Borovička (2015) for heterogeneous beliefs and i.i.d. consumption. Furthermore we extend the analysis in Gârleanu and Panageas (2014) (who rely on an overlapping generations model and thus on 'survival by construction') by showing that survival of both investors can also arise endogenously.

4.2 Long-Run Growth Risk

We now analyze the model including the long-run growth rate X as a state variable. For the numerical analyses below we assume $\overline{\mu}_C = 0.02$, $\sigma_C = 0.0252$, $\beta_1 = \beta_2 = 0.10$, $\gamma_1 = 4$, and $\gamma_2 = 10$. Furthermore, we set $\kappa_x = 0.3$ and $\sigma_x = 0.0114$. As in the case without a state variable, we then vary ψ_1 and ψ_2. To study the impact of heterogeneity with respect to risk aversion, we compare the base case also to a specification with $\gamma_1 = 6$ and $\gamma_2 = 8$.

Since we are in an economy with EZ preferences, X-risk will be priced. This positive risk premium (in case of a preference for early resolution of uncertainty) will have an impact on the investors' portfolio and savings decisions, and thus also on the long-run evolution of the investors' respective consumption shares.

In terms of solving the model there are no longer closed-form solutions for the coefficients of the stochastic processes for ω, and thus also no longer a survival condition which can be represented in closed form. We will instead resort to Monte Carlo simulations, where we track the evolution of consumption shares over a horizon of $T = 500$ years along 1,000 paths with monthly time steps. The initial value of the state variable X is set equal to its long-run mean of zero.

The idea behind our analysis is the following: We look at the investors' average consumption shares at the very distant future point in time $T = 500$, starting from a point where one investor is small (and, consequently, the other one is large). For investor 1 being initially small is represented by initial consumption share of $\omega_0 = 0.1$, and for investor 2 it is in a symmetric fashion given by an initial share of $1 - \omega_0 = 0.1$. If then for a certain combination of ψ_1 and ψ_2 both investors' average consumption shares increase from a point where they are small initially, we take this as evidence that both investors retain a non-negligible (average) consumption share over the long run. If on the other hand one initially small investor's average consumption share grows up to $T = 500$, while for the other one it decreases, this represents a case where we cannot make a strong statement concerning the evolution of consumption shares in the limit. It might be the case that the investors' consumption shares converge towards a distribution where one investor retains, e.g., 5% and the other one 95% of aggregate consumption, but it might also be that one investor actually vanishes.

Table 1 shows the results of this analysis for $\omega_0 = 0.1$ (upper panel) and $\omega_0 = 0.9$ (lower panel) under the base case parametrization with $\gamma_1 = 4$ and $\gamma_2 = 10$.

The numbers in the table make sense intuitively. Across a given row in the table investor 1's average consumption share is monotonically increasing in ψ_1, since investor 1 is in a better and better position relative to investor 2 with respect to the savings channel. Analogously, in a given column investor 1's average share is decreasing in ψ_2.

The question of key interest is now whether for a given combination of ψ_1 and ψ_2 ω_{500} is greater than 0.1 in the upper and less than 0.9 in the lower panel of the table, since these scenarios are those that we identify with increasing average consumption shares for initially small investors.

An example is the case $\psi_1 = 0.4$ and $\psi_2 = 0.5$. Starting from a level of $\omega_0 = 0.1$, investor 1's average consumption share has increased to around 0.37 after 500 years, as shown in the upper panel of the table. When her initial share is 0.9 on the other hand, it reduces on average to about 0.77 at $T = 500$. This in turn implies that investor 2's share has gone up from 0.1 to around 0.23 on average. So both investors exhibit increasing average consumption shares when small initially. These cases are marked by the entry '1/2' in Table 2, whereas '1' ('2') indicate the cases when investor 1's (2's) average consumption share is increasing from an initially low level.

Looking at the overall location of the '1/2' entries in Table 2, we find that not surprisingly they are found below the main diagonal, i.e., in cases when $\psi_2 > \psi_1$. The reason for this is that investor 1 has the advantage with respect to risk aversion, so that she will tend to hold the more aggressive portfolio, which lets her earn a higher risk premium. This has to be compensated from the perspective of investor 2 by a higher EIS, so that she can save more to make her consumption share grow on average.

The numbers in the table also give an idea of how much the relative advantage with respect to the EIS must be to not just compensate but actually over-compensate the disadvantage from a higher risk aversion. With our parametrization of the consumption and growth rate processes, investor 2 has a chance to take over the economy only for the lowest value of $\psi_1 = 0.4$ considered in our analysis, and there also only with a markedly higher ψ_2 (in our case for $\psi_2 \geq 1.1$).[10]

We vary the base case with respect to two main features. One is the difference between the two investors in their degrees of risk aversion. Here we consider $\gamma_1 = 6$ and $\gamma_2 = 8$ as an alternative scenario, i.e., a case where the two investors are more similar than in the base case. The second dimension of variation is consumption dynamics. We have already analyzed the case of i.i.d. consumption growth above, but to asses how much it matters, if there is actually long-run risk, we perform our simulation analyses also for that setting.

With respect to differences in risk aversion, a comparison of Tables 2 and 4 shows that with a smaller difference between γ_1 and γ_2, differences in the EIS become more relevant with respect to the long-run evolution of average consumption shares. The '1/2'-region is considerably smaller than in the base case. Even small advantages with respect to the EIS can now (but again need not) lead to the respective investor being the only one with an increasing average consumption share in the long run. The intuition behind these results is that starting from $\omega_0 = 0.1$ investor 1 now has a smaller advantage with respect to risk aversion than in the base case. Consequently, if it grows at all, her average consumption share will grow at a lower rate, or, in the opposite case, will decrease faster. Furthermore, there are many combinations of ψ_1 and ψ_2 for which ω no longer grows but falls on average. For investor 2 we can exactly mirror this argument, i.e., her disadvantage with respect to γ is now less pronounced, so that, if her average consumption share grows, it will grow faster, or, in the opposite case, decrease more slowly.

[10] Again, we cannot rule out the case that the investors' average consumption shares converge to values like 0.05 for investor 1 and 0.95 for investor 2.

When we then look at the impact of the presence of long-run growth risk as compared to the i.i.d. case, we see that the 1/2-region is now larger than without a state variable. Comparing Tables 2 and 6 we find that it only takes a much smaller advantage with respect to EIS for investor 2 to become the only one with an increasing average consumption share. This is due to that fact that now, without long-run risk, there is a smaller risk premium to be picked up, so that the advantage with respect to risk aversion is less relevant than in a situation with X-risk present. In a sense this result is analogous to the one obtained before for a smaller difference between γ_1 and γ_2. The fundamental issue is that whenever risk premia become smaller or less relevant, be it due to more similar levels of risk aversion or smaller risk premia in general, the region of ψ-combinations with increasing average consumption shares for both investors becomes smaller.

Finally, we take a look at the correlation between consumption shares and the evolution of the expected growth rate X. As discussed above, there is no closed-form condition for survival in the case with a state variable, and one of the main reasons for that is that ω and X are potentially correlated.

Tables 9 and 10 show estimates of the correlation between ω_{500} and X_{500}, i.e., between investor 1's consumption share and the long-run growth rate, both measured at $T = 500$. The evidence is clear. In those cases where investor 1 enjoys the double advantage of lower risk aversion and higher EIS, the estimated correlations are indeed small in absolute value (the fact that the sign sometimes switches is likely due to simulation error). This makes sense, since in these cases there is no doubt that investor 1 will dominate in the long run, and her consumption share will be large basically no matter how X has evolved. In all other cases, where either investor 2 is the only one with an increasing average consumption share or where both investors grow on average when they are small initially, the correlations are positive and substantially away from zero.

5 Conclusion

In this paper we have analyzed an economy with two heterogeneous EZ investors and stochastic long-run growth risk. The focus of our analysis was on the conditions under which both investors can survive in the long run (or at least exhibit increasing average consumption shares over the long term when they are small initially). Since many analyses in asset pricing models are performed for the steady state of the economy, the long-run presence of both investors is crucial for a meaningful analysis of a heterogeneous-agent economy.

For the analysis of long-run survival the evolution of the small investor's consumption share near zero is crucial. We have identified two channels via which the small investor can actually avoid extinction. One is to 'save his way out' (as shown by Borovička (2015)) by having a high enough wealth-consumption ratio relative to the large investor. The other is to 'speculate his way out of extinction' by holding a riskier portfolio with a higher expected excess return. The first strategy is followed by the investor with the higher EIS, while the second is adopted by the investor with the lower risk aversion.

For the case of i.i.d. consumption we can solve for the survival condition in closed form. Our main interest is in the impact of the combinations of preference parameters, and we find that, in contrast to the CRRA case, EZ preferences create an *area* (not jsut a line) of combinations of preference parameters for which both investors can survive in the long run. This result complements the findings of Borovička (2015), who studies a model where agents have identical EZ preferences, but differ in their beliefs.

We extend his analysis further by studying the long-run evolution of (average) consumption shares also for the case when consumption growth is not i.i.d., i.e., when the expected growth rate is stochastic. Here the main finding is that the area of preference parameter combinations for which both investors retain non-negligible average consumption shares in the long run is even larger than in the i.i.d. case. The reason is that the survival becomes 'easier' (compared to the i.i.d framework) for the small investor when she exhibits a low degree of risk aversion. She will then be willing to hold a riskier portfolio, and the additional risk premium on the long-run growth factor contributes significantly to the overall expected excess return on her wealth. The savings channel on the other hand is hardly affected by the introduction of a state variable.

A Appendix

A.1 Assumptions

Our key assumptions are that when an investor becomes large the partial derivative of her wealth-consumption ratio with respect to ω remains finite, and that when she becomes small this partial derivative multiplied by her consumption share goes to zero, i.e., $\lim_{\omega\to 1}\frac{\partial v_1}{\partial\omega} < \infty$ and $\lim_{\omega\to 0}\frac{\partial v_1}{\partial\omega}\,\omega = 0$ (see Borovička (2015)). Furthermore, any derivatives of the wealth-consumption ratios with respect to x are assumed to be finite. Throughout this appendix, we consider the case with a state variable. The respective formulas for the economy without a state variable follow as special cases, but are not given explicitly.

A.2 Properties of Consumption and Wealth-Consumption Ratios

A.2.1 Volatility of Consumption

The volatility of investor 1's consumption is given by $\sigma_{C_1} = \sigma_C + \frac{1}{\omega}\sigma_\omega$. The general expression for the volatility of the consumption share is given in Equation (9). For $\omega\to 0$ the limiting value of the normalized volatility is

$$\lim_{\omega\to 0}\frac{1}{\omega}\sigma_\omega \;=\; \frac{\left(\frac{\gamma_2}{\gamma_1}-1\right)\sigma_C + \frac{1}{\gamma_1}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X}{1+\frac{1-\theta_1}{\gamma_1}\lim\limits_{\omega\to 0}\frac{\partial v_1}{\partial\omega}\,\omega}.$$

Under the assumption that $\lim\limits_{\omega\to 0}\frac{\partial v_1}{\partial\omega}\,\omega\to 0$, this yields

$$\lim_{\omega\to 0}\frac{1}{\omega}\sigma_\omega \;=\; \left(\frac{\gamma_2}{\gamma_1}-1\right)\sigma_C + \frac{1}{\gamma_1}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X.$$

From Equation (9) we obtain $\lim_{\omega\to 1}\frac{1}{\omega}\sigma_\omega = 0$, implying

$$\lim_{\omega\to 1}\sigma_{C_1} = \sigma_C$$

and

$$\lim_{\omega\to 0}\sigma_{C_1} = \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X.$$

A.2.2 Volatility of Wealth-Consumption Ratio

Given $v_1 \equiv v_1(\omega, x)$ the volatility of the log wealth-consumption ratio is

$$\sigma_{v_1} \;=\; \frac{\partial v_1}{\partial\omega}\sigma_\omega + \frac{\partial v_1}{\partial x}\sigma_X.$$

$\frac{\partial v_1}{\partial \omega}$ remains bounded for $\omega \to 1$ by assumption and the volatility of the consumption share goes to zero (as shown above), so that $\frac{\partial v_1}{\partial \omega}\sigma_\omega$ also goes to zero. This yields

$$\lim_{\omega \to 1} \sigma_{v_1} = \frac{\partial v_1}{\partial x}\sigma_X.$$

For the small investor we also obtain

$$\lim_{\omega \to 0} \sigma_{v_1} = \frac{\partial v_1}{\partial x}\sigma_X.$$

To see this note that

$$\frac{\partial v_1}{\partial \omega}\sigma_\omega = \left(\frac{\partial v_1}{\partial \omega}\omega\right)\left(\frac{1}{\omega}\sigma_\omega\right),$$

where the first term is zero by assumption and the second remains finite as shown above, so that product goes to zero.

A.2.3 Drift of Wealth-Consumption Ratio

The drift of the log wealth-consumption ratio is

$$\begin{aligned}
\mu_{v_1} &= \frac{\partial v_1}{\partial \omega}\mu_\omega - \frac{\partial v_1}{\partial x}\kappa X_t + 0.5\frac{\partial^2 v_1}{\partial \omega^2}\sigma_\omega'\sigma_\omega + 0.5\frac{\partial^2 v_1}{\partial x^2}\sigma_X'\sigma_X + \frac{\partial^2 v_1}{\partial \omega \partial x}\sigma_\omega'\sigma_X \\
&= \frac{\partial v_1}{\partial \omega}\omega\,\frac{1}{\omega}\mu_\omega - \frac{\partial v_1}{\partial x}\kappa X_t \\
&\quad + 0.5\frac{\partial^2 v_1}{\partial \omega^2}\omega^2\,\frac{1}{\omega}\sigma_\omega'\frac{1}{\omega}\sigma_\omega + 0.5\frac{\partial^2 v_1}{\partial x^2}\sigma_X'\sigma_X + \frac{\partial^2 v_1}{\partial \omega \partial x}\omega\,\frac{1}{\omega}\sigma_\omega'\sigma_X.
\end{aligned}$$

When investor 1 is large, the normalized drift and volatility of the consumption share converge to zero. This yields

$$\lim_{\omega \to 1} \mu_{v_1} = -\frac{\partial v_1}{\partial x}\kappa X_t + 0.5\frac{\partial^2 v_1}{\partial x^2}\sigma_X'\sigma_X.$$

When investor 1 is small, $\frac{\partial v_1}{\partial \omega}\omega$ and $\frac{\partial^2 v_1}{\partial \omega^2}\omega^2$ are zero by assumption. The normalized drift and volatility of the consumption share then converge to some finite number, and we get

$$\lim_{\omega \to 0} \mu_{v_1} = -\frac{\partial v_1}{\partial x}\kappa X_t + 0.5\frac{\partial^2 v_1}{\partial x^2}\sigma_X'\sigma_X.$$

So at the boundaries the drift of the log wealth-consumption ratio is independent of the consumption share ω.

A.2.4 Partial Differential Equation for Wealth-Consumption Ratio

The partial differential equation for the log wealth-consumption ratio v_i is

$$\begin{aligned} 0 &= e^{-v_1} - \beta_1 + \left(1 - \frac{1}{\psi_1}\right)\left(\mu_{C_1} - 0.5\gamma_1 \sigma_{C_1}' \sigma_{C_1}\right) \\ &\quad + \mu_{v_1} + 0.5\theta_1 \sigma_{v_1}' \sigma_{v_1} + (1-\gamma_1)\sigma_{C_1}' \sigma_{v_1}. \end{aligned}$$

Plugging in the volatility and the drift of the log wealth-consumption ratio for the large investor from Appendices A.2.2 and A.2.3 we obtain

$$\begin{aligned} 0 &= e^{-v_1} - \beta_1 + \left(1 - \frac{1}{\psi_1}\right)\left(\mu_C + X_t - 0.5\gamma_1 \sigma_C' \sigma_C\right) \\ &\quad - \frac{\partial v_1}{\partial x}\kappa X_t + 0.5\frac{\partial^2 v_1}{\partial x^2}\sigma_X' \sigma_X + 0.5\theta_1 \left(\frac{\partial v_1}{\partial x}\right)^2 \sigma_X' \sigma_X + (1-\gamma_1)\sigma_C' \frac{\partial v_1}{\partial x}\sigma_X. \end{aligned}$$

This equation coincides with the partial differential equation in an economy with one investor only, and the dependence on the consumption share vanishes.

For the small investor the partial differential equation becomes

$$\begin{aligned} 0 &= e^{-v_1} - \psi_1\beta_1 - (1-\psi_1)\beta_2 + (\psi_1 - 1)\frac{1}{\psi_2}(\mu_C + X_t) \\ &\quad + (\psi_1 - 1)\left\{ -0.5\gamma_2\left(1 + \frac{1}{\psi_2}\right)\sigma_C'\sigma_C - 0.5(1-\theta_2)\sigma_{v_2}'\sigma_{v_2} - (1-\theta_2)\sigma_C'\sigma_{v_2} \right\} \\ &\quad + 0.5\gamma_1(\psi_1 - 1)\left\{ \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1\gamma_2}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X \right\}' \\ &\qquad\qquad \left\{ \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1\gamma_2}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X \right\} \\ &\quad + 0.5\left[(\psi_1 - 1)(1-\theta_1) + \theta_1\right]\left(\frac{\partial v_1}{\partial x}\right)^2 \sigma_X'\sigma_X \\ &\quad + \gamma_1(\psi_1 - 1)\left\{ \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1\gamma_2}\left[(1-\theta_2)\frac{\partial v_2}{\partial x} - (1-\theta_1)\frac{\partial v_1}{\partial x}\right]\sigma_X \right\}' \frac{\partial v_1}{\partial x}\sigma_X \\ &\quad - \frac{\partial v_1}{\partial x}\kappa X_t + 0.5\frac{\partial^2 v_1}{\partial x^2}\sigma_X'\sigma_X. \end{aligned}$$

The dependence on the consumption share has vanished. Note that the function v_2 and its partial derivatives have already been solved for and are known when it comes to finding $\lim_{\omega \to 0} v_1(\omega, x)$. For the latter function, we are thus left with an ordinary differential equation in x.

A.3 Approximations

A.3.1 Approximating the Derivatives of the Log Wealth-Consumption Ratio

Differentiating both sides of (6) with respect to x and assuming that the volatilities are independent of x and the second derivative of v_i with respect to x is zero one obtains

$$\begin{aligned} 0 &\approx -e^{-v_i}\frac{\partial v_i}{\partial x} + \left(1 - \frac{1}{\psi_i}\right)\frac{\partial \mu_{C_1}}{\partial x} - \frac{\partial v_i}{\partial x}\kappa_x \\ \Rightarrow \frac{\partial v_i}{\partial x} &\approx \frac{1 - \frac{1}{\psi_i}}{e^{-v_i} + \kappa_x} \cdot \frac{\partial \mu_{C_i}}{\partial x}. \end{aligned} \tag{A.1}$$

We now need to find $\frac{\partial \mu_{C_i}}{\partial x}$. Taking Equation (2) and plugging in σ_ω and μ_ω from Equations (9) and (10) yields

$$\begin{aligned} \mu_{C_1} &= \frac{\psi_1}{\bar{\psi}}\left(\mu_C + X_t\right) \\ &\quad + \frac{(1-\omega)\psi_1\psi_2}{\bar{\psi}}\Bigg\{\beta_2 - \frac{\gamma_2}{2}\left(1 + \frac{1}{\psi_2}\right)\sigma'_{C_2}\sigma_{C_2} - \frac{(1-\theta_2)}{2}\sigma'_{v_2}\sigma_{v_2} - (1-\theta_2)\sigma'_{C_2}\sigma_{v_2} \\ &\qquad - \beta_1 + \frac{\gamma_1}{2}\left(1 + \frac{1}{\psi_1}\right)\sigma'_{C_1}\sigma_{C_1} + \frac{(1-\theta_1)}{2}\sigma'_{v_1}\sigma_{v_1} + (1-\theta_1)\sigma'_{C_1}\sigma_{v_1}\Bigg\}. \end{aligned}$$

Differentiating both sides with respect to x and assuming that all volatility terms are independent of x we get

$$\frac{\partial \mu_{C_1}}{\partial x} \approx \frac{\psi_1}{\bar{\psi}}.$$

For the relevant limits this implies

$$\begin{aligned} \lim_{\omega\to 0}\frac{\partial \mu_{C_1}}{\partial x} &\approx \frac{\psi_1}{\psi_2} \\ \lim_{\omega\to 1}\frac{\partial \mu_{C_1}}{\partial x} &\approx 1, \end{aligned}$$

since the average $\bar{\psi}$ is equal to the large investor's EIS. Plugging this into (A.1) gives

$$\lim_{\omega\to 1}\frac{\partial v_1}{\partial x} \approx \frac{1 - \frac{1}{\psi_1}}{e^{-v_1} + \kappa_x}.$$

and

$$\lim_{\omega\to 0}\frac{\partial v_1}{\partial x} \approx \frac{1 - \frac{1}{\psi_1}}{e^{-v_1} + \kappa_x} \cdot \frac{\psi_1}{\psi_2}.$$

A.3.2 Approximating the Volatility of Individual Consumption

With the limiting approximations for the derivatives of the log wealth-consumption ratio derived above, we can now determine approximate expressions for the volatility of consumption. When investor 1 is large, it obviously holds that $\lim_{\omega \to 1} \sigma_{C_1} = \sigma_C$.

For the small investor we obtain

$$\lim_{\omega \to 0} \sigma_{C_1} \approx \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1}\left[\frac{\gamma_2 - \frac{1}{\psi_2}}{e^{-v_2} + \kappa_x} - \frac{\gamma_1 - \frac{1}{\psi_1}}{e^{-v_1} + \kappa_x} \cdot \frac{\psi_1}{\psi_2}\right]\sigma_X.$$

To further simplify this expression we ignore potential differences between v_1 and v_2 and set $e^{v_1} \approx e^{v_2} \approx e^{\bar{v}}$, yielding

$$\lim_{\omega \to 0} \sigma_{C_1} \approx \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1 \psi_2} \cdot \frac{\gamma_2\psi_2 - \gamma_1\psi_1}{e^{-\bar{v}(0)} + \kappa_x}\sigma_X.$$

A.3.3 Approximating the Volatility of Wealth

Equipped with the limiting values for the dependence of the wealth-consumption ratio on x from Appendix A.3.1, we can now determine approximate expressions for the volatility of wealth. If investor 1 is large, we obtain

$$\lim_{\omega \to 1} \sigma_{V_1} \approx \sigma_C + \frac{1 - \frac{1}{\psi_1}}{e^{-v_1} + \kappa_x}\sigma_X.$$

For the small investor and under the additional assumption that $\lim_{\omega \to 0} \frac{\partial v_1}{\partial \omega}\,\omega \to 0$, the limiting value for the volatility of wealth is

$$\lim_{\omega \to 0} \sigma_{V_1} \approx \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{1}{\gamma_1}\left[\frac{\gamma_2 - \frac{1}{\psi_2}}{e^{-v_2} + \kappa_x} - \frac{\gamma_1 - \frac{1}{\psi_1}}{e^{-v_1} + \kappa_x} \cdot \frac{\psi_1}{\psi_2}\right]\sigma_X + \frac{1 - \frac{1}{\psi_1}}{e^{-v_1} + \kappa_x}\frac{\psi_1}{\psi_2}\sigma_X.$$

To further simplify this term, we ignore any differences in the level of wealth-consumption ratios and set $e^{-v_1} \approx e^{-v_2} \approx e^{-\bar{v}}$ for $\omega \to 0$, which finally gives

$$\lim_{\omega \to 0} \sigma_{V_1} \approx \frac{\gamma_2}{\gamma_1}\sigma_C + \frac{\frac{\gamma_2}{\gamma_1} - \frac{1}{\psi_2}}{e^{-\bar{v}} + \kappa_x}\sigma_X.$$

A.4 Behavior of the Consumption Share at the Boundaries

The long-run survival of the investors depends on the behavior of the consumption share at the boundaries $\omega = 0$ (where investor 1 is small) and $\omega = 1$ (where investor 2 is small). The exposition and analysis here closely follows Borovička (2015).

If the left boundary $\omega = 0$ is attracting, investor 2 will take over the whole economy when investor 1's consumption becomes small. In the opposite case, investor 1 can survive. A similar analysis holds for the right boundary. This results in four different cases:

1. If both boundaries are not attracting, neither investor 1 nor investor 2 will dominate in the long run, but both investors can survive.

2. If the left boundary $\omega = 0$ is attracting while the right boundary $\omega = 1$ is not, investor 2 will dominate in the long run.

3. If the left boundary $\omega = 0$ is not attracting while the right boundary $\omega = 1$ is, investor 1 will dominate in the long run.

4. If both boundaries are attracting, investor 1 and investor 2 will dominate the economy with a positive probability each.

We present the analysis for the case without a state variable. Following Karlin and Taylor (1981) and Borovička (2015) the condition for the left boundary not to be attracting is that

$$\lim_{\omega\to 0} \tilde{\mu}_\omega(\omega) - \frac{1}{2}\tilde{\sigma}^2_\omega(\omega) > 0, \tag{A.2}$$

where

$$\begin{aligned}\tilde{\mu}_\omega(\omega) &= \frac{\mu_\omega(\omega)}{\omega(1-\omega)}\\ \tilde{\sigma}_\omega(\omega) &= \frac{\sigma_\omega(\omega)}{\omega(1-\omega)},\end{aligned}$$

and $\mu_\omega(\omega)$ and $\sigma_\omega(\omega)$ are the drift and the volatility of the consumption share ω, i.e.

$$d\omega = \mu_\omega(\omega)dt + \sigma_\omega(\omega)'dW.$$

Intuitively, condition (A.2) says that the drift of the log of ω is positive in the limit as $\omega \to 0$, or, equivalently, $\ln\omega \to -\infty$. From Ito

$$d\ln\omega = \left(\frac{1}{\omega}\mu_\omega(\omega) - \frac{1}{2\omega^2}\sigma^2_\omega(\omega)\right) dt + \frac{1}{\omega}\sigma_\omega(\omega)'dW,$$

In the limit as $\omega \to 0$ (and thus $(1-\omega) \to 1$) $\frac{1}{\omega}\mu_\omega(\omega)$ is just equal to $\tilde{\mu}_\omega(\omega)$. An analogous statement holds for $\frac{1}{\omega}\sigma_\omega(\omega)$ and $\tilde{\sigma}_\omega(\omega)$. A positive drift for $\ln\omega$ at the left boundary is thus equivalent to condition (A.2). The division by $1-\omega$ in the definition of $\tilde{\mu}_\omega(\omega)$ and $\tilde{\sigma}_\omega(\omega)$ is done for reasons of symmetry between the cases $\omega \to 0$ and $\omega \to 1$, since the dynamics of $\ln(1-\omega)$ have the same structure as those for $\ln\omega$, with $1-\omega$ replacing ω everywhere.

For the case of EZ preferences in the model without a state variable, the left boundary is not attracting when Condition (12) holds. This condition is obtained by substituting for $\tilde{\mu}_\omega(\omega)$ and $\tilde{\sigma}_\omega(\omega)$ in (A.2), using the expressions for the drift and the volatility of ω derived above (where all terms related to x drop out).

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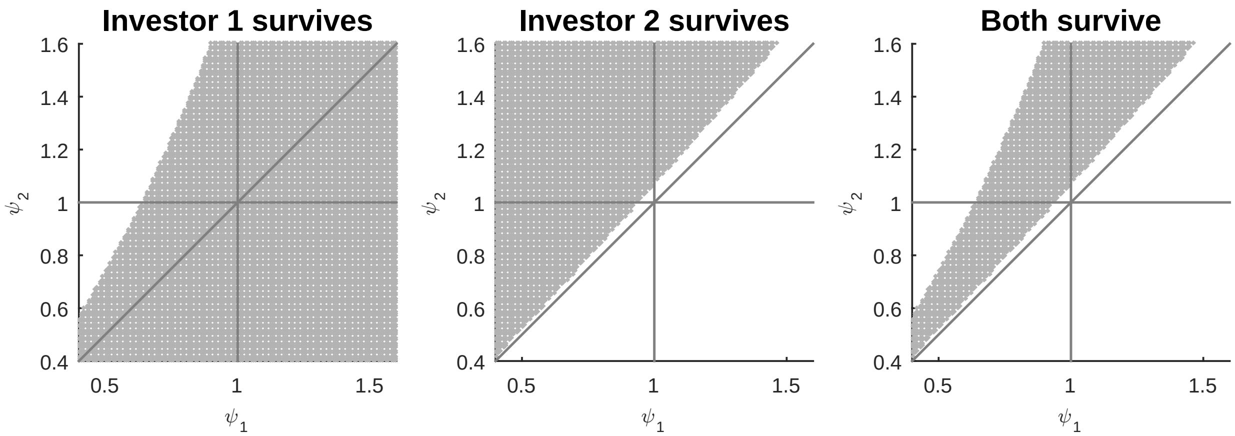

Figure 1: Long-run survival (i.i.d. consumption growth)

The figure shows the combinations of the intertemporal elasticity of substitution of investor 1 (x-axis) and investor 2 (y-axis) for which investor 1 (left graph), investor 2 (middle graph), and both investors (right graph) survive. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.1$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$.

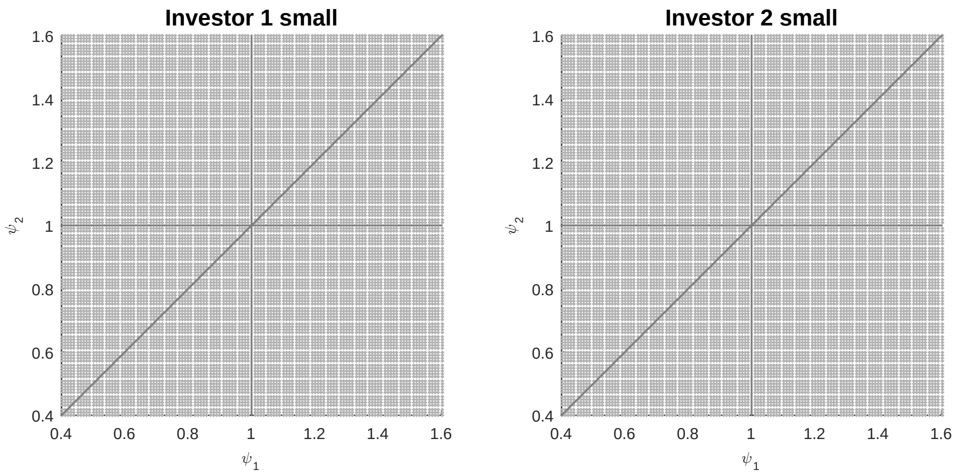

Figure 2: Bounded wealth-consumption ratio of small investor (i.i.d. consumption growth)

The figure shows the combinations of the intertemporal elasticity of substitution of investor 1 (x-axis) and investor 2 (y-axis) for which the wealth-consumption ratio of the small investor is bounded. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.1$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$.

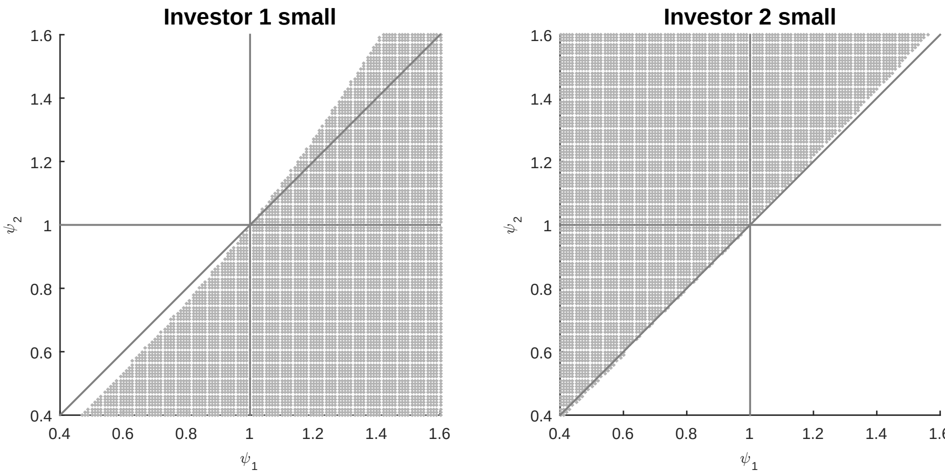

Figure 3: Small investor saves more (i.i.d. consumption growth)

The figure shows the combinations of the EIS of investor 1 (x-axis) and investor 2 (y-axis) for which the wealth-consumption ratio of the small investor exceeds the wealth-consumption ratio of the large investor, i.e. for which the small investor saves more. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.1$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$.

$\omega_0 = 0.10$

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.58127	0.83606	0.93768	0.97295	0.98660	0.99204	0.99661	0.99746	0.99807	0.99845	0.99872	0.99892
0.50	0.36867	0.62136	0.82076	0.92427	0.96232	0.98027	0.99260	0.99493	0.99631	0.99721	0.99778	0.99824
0.60	0.25327	0.44290	0.65403	0.81992	0.91075	0.95376	0.98416	0.98984	0.99300	0.99491	0.99617	0.99710
0.70	0.18685	0.32775	0.51407	0.69407	0.82767	0.90658	0.96828	0.98055	0.98752	0.99114	0.99356	0.99504
0.80	0.15190	0.26243	0.40112	0.56272	0.71932	0.83426	0.94393	0.96395	0.97832	0.98465	0.98923	0.99212
0.90	0.12925	0.21925	0.32751	0.46419	0.60813	0.74286	0.90380	0.94069	0.96172	0.97485	0.98285	0.98743
1.10	0.09979	0.16269	0.24064	0.33869	0.44692	0.57378	0.78545	0.86085	0.90868	0.93937	0.96028	0.97127
1.20	0.09245	0.14254	0.20999	0.29569	0.39243	0.50168	0.71670	0.80845	0.86668	0.91257	0.93983	0.95798
1.30	0.08442	0.13138	0.19973	0.26846	0.35245	0.45346	0.65719	0.74498	0.81846	0.87729	0.91383	0.94002
1.40	0.07523	0.12111	0.17745	0.24396	0.31751	0.40738	0.60506	0.68828	0.77297	0.83640	0.88424	0.92092
1.50	0.07160	0.11374	0.16416	0.22616	0.29468	0.37429	0.54410	0.63432	0.72051	0.79157	0.84835	0.89489
1.60	0.06818	0.10846	0.15190	0.21176	0.27330	0.34025	0.50431	0.59146	0.66986	0.74907	0.81230	0.86353

$\omega_0 = 0.90$

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.95393	0.99126	0.99714	0.99879	0.99937	0.99961	0.99981	0.99985	0.99988	0.99990	0.99991	0.99993
0.50	0.77278	0.95776	0.98929	0.99628	0.99823	0.99906	0.99961	0.99972	0.99978	0.99983	0.99986	0.99989
0.60	0.50485	0.83744	0.96123	0.98850	0.99525	0.99773	0.99921	0.99947	0.99963	0.99972	0.99978	0.99983
0.70	0.34339	0.64051	0.88261	0.96532	0.98764	0.99465	0.99838	0.99899	0.99933	0.99951	0.99964	0.99973
0.80	0.26361	0.45955	0.72763	0.90850	0.96844	0.98741	0.99677	0.99815	0.99879	0.99917	0.99941	0.99957
0.90	0.21547	0.35096	0.57131	0.80151	0.92471	0.97069	0.99363	0.99652	0.99783	0.99863	0.99903	0.99929
1.10	0.16261	0.24826	0.36945	0.54351	0.73721	0.87849	0.97618	0.98787	0.99317	0.99599	0.99739	0.99829
1.20	0.14584	0.21925	0.31483	0.45472	0.62633	0.79279	0.95617	0.97851	0.98832	0.99332	0.99594	0.99734
1.30	0.13057	0.19919	0.27833	0.38903	0.53339	0.69867	0.91870	0.96187	0.98047	0.98932	0.99364	0.99586
1.40	0.12263	0.18320	0.24948	0.34677	0.45890	0.60829	0.86785	0.93332	0.96747	0.98192	0.98977	0.99358
1.50	0.11767	0.16789	0.23184	0.31300	0.40777	0.52749	0.80287	0.89687	0.94540	0.97096	0.98373	0.99024
1.60	0.11158	0.15281	0.21532	0.28285	0.36733	0.47679	0.73365	0.84066	0.91433	0.95393	0.97404	0.98451

Table 1: Long-term average consumption share of investor 1 ($T = 500$, $\gamma_1 = 4$, $\gamma_2 = 10$)

The table gives the average consumption share of investor 1 after 500 years. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is $\omega_0 \in \{0.1, 0.9\}$, the initial value of the state variable x is set to zero. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	1	1	1	1	1	1	1	1	1	1	1	1
0.50	1/2	1	1	1	1	1	1	1	1	1	1	1
0.60	1/2	1/2	1	1	1	1	1	1	1	1	1	1
0.70	1/2	1/2	1/2	1	1	1	1	1	1	1	1	1
0.80	1/2	1/2	1/2	1	1	1	1	1	1	1	1	1
0.90	1/2	1/2	1/2	1/2	1	1	1	1	1	1	1	1
1.10	2	1/2	1/2	1/2	1/2	1/2	1	1	1	1	1	1
1.20	2	1/2	1/2	1/2	1/2	1/2	1	1	1	1	1	1
1.30	2	1/2	1/2	1/2	1/2	1/2	1	1	1	1	1	1
1.40	2	1/2	1/2	1/2	1/2	1/2	1/2	1	1	1	1	1
1.50	2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1	1	1	1
1.60	2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1	1	1	1

Table 2: Long-run increasing average consumption shares ($T = 500$, $\gamma_1 = 4$, $\gamma_2 = 10$)

The entries in the table indicate if the average consumption share of (an initially small) investor 1 and/or (an initially small) investor 2 have increased until $T = 500$ relative to the respective starting values of 0.1. '1', '2', and '1/2' indicate that investor 1's, 2's, or both investors' respective consumption share has increased on average. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is $\omega_0 \in \{0.1, 0.9\}$, the initial value of the state variable X is set to zero. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.

$\omega_0 = 0.10$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.22074	0.61152	0.85254	0.93797	0.96866	0.98203	0.99204	0.99418	0.99548	0.99640	0.99704	0.99751
0.50	0.06250	0.23850	0.55166	0.79474	0.90346	0.94983	0.98139	0.98738	0.99067	0.99303	0.99447	0.99551
0.60	0.02475	0.08723	0.25655	0.52135	0.74881	0.87001	0.95714	0.97232	0.98121	0.98631	0.98984	0.99203
0.70	0.01268	0.03869	0.11002	0.27098	0.50700	0.71078	0.90591	0.94127	0.96218	0.97385	0.98115	0.98603
0.80	0.00759	0.02100	0.05571	0.13561	0.28873	0.49515	0.81075	0.88716	0.92771	0.95170	0.96679	0.97576
0.90	0.00510	0.01290	0.03118	0.07266	0.15923	0.30885	0.66484	0.78843	0.86491	0.91214	0.94150	0.95821
1.10	0.00290	0.00615	0.01335	0.02804	0.05723	0.11283	0.34081	0.49955	0.64740	0.75610	0.83639	0.88727
1.20	0.00233	0.00470	0.00969	0.01922	0.03806	0.07228	0.23119	0.35919	0.50279	0.63960	0.74877	0.82365
1.30	0.00193	0.00377	0.00730	0.01399	0.02649	0.04926	0.15527	0.25569	0.38243	0.51473	0.63964	0.73985
1.40	0.00166	0.00310	0.00570	0.01068	0.01928	0.03539	0.10776	0.18139	0.27781	0.39690	0.52368	0.64317
1.50	0.00145	0.00261	0.00471	0.00837	0.01503	0.02644	0.07814	0.12847	0.20291	0.29933	0.41416	0.53302
1.60	0.00128	0.00226	0.00390	0.00679	0.01194	0.02023	0.05782	0.09300	0.14920	0.22497	0.32419	0.43157
$\omega_0 = 0.90$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.94123	0.98770	0.99581	0.99808	0.99889	0.99929	0.99963	0.99971	0.99976	0.99980	0.99983	0.99985
0.50	0.72415	0.94433	0.98488	0.99415	0.99715	0.99836	0.99928	0.99945	0.99958	0.99967	0.99973	0.99977
0.60	0.39646	0.79693	0.94797	0.98250	0.99256	0.99620	0.99856	0.99900	0.99927	0.99944	0.99955	0.99963
0.70	0.20531	0.53965	0.84274	0.95024	0.98129	0.99129	0.99720	0.99815	0.99873	0.99907	0.99927	0.99943
0.80	0.11791	0.32189	0.64641	0.87111	0.95365	0.98016	0.99441	0.99660	0.99777	0.99844	0.99884	0.99911
0.90	0.07534	0.19588	0.43876	0.72398	0.89173	0.95591	0.98930	0.99365	0.99611	0.99734	0.99814	0.99861
1.10	0.03764	0.08897	0.19529	0.38484	0.62805	0.82127	0.96051	0.97905	0.98811	0.99251	0.99507	0.99660
1.20	0.02923	0.06553	0.13827	0.27309	0.47822	0.69808	0.92713	0.96277	0.97933	0.98756	0.99220	0.99473
1.30	0.02347	0.05000	0.10209	0.19911	0.35541	0.55931	0.87207	0.93457	0.96449	0.97950	0.98753	0.99181
1.40	0.01957	0.03939	0.07841	0.14837	0.26610	0.43679	0.79361	0.88922	0.94105	0.96648	0.97996	0.98733
1.50	0.01639	0.03207	0.06156	0.11386	0.20331	0.34022	0.69374	0.82674	0.90547	0.94616	0.96833	0.98033
1.60	0.01437	0.02734	0.05039	0.09025	0.15917	0.26469	0.58499	0.74290	0.85154	0.91582	0.95078	0.97029

Table 3: Long-term average consumption share of investor 1 ($T = 500$, $\gamma_1 = 6$, $\gamma_2 = 8$)

The table gives the average consumption share of investor 1 after 500 years. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is $\omega_0 \in \{0.1, 0.9\}$, the initial value of the state variable x is set to zero. The parameters of relative risk aversion are $\gamma_1 = 6$ and $\gamma_2 = 8$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.

$\downarrow \psi_2$ | $\psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	1	1	1	1	1	1	1	1	1	1	1	1
0.50	2	1	1	1	1	1	1	1	1	1	1	1
0.60	2	2	1	1	1	1	1	1	1	1	1	1
0.70	2	2	1/2	1	1	1	1	1	1	1	1	1
0.80	2	2	2	1/2	1	1	1	1	1	1	1	1
0.90	2	2	2	2	1/2	1	1	1	1	1	1	1
1.10	2	2	2	2	2	1/2	1	1	1	1	1	1
1.20	2	2	2	2	2	2	1	1	1	1	1	1
1.30	2	2	2	2	2	2	1/2	1	1	1	1	1
1.40	2	2	2	2	2	2	1/2	1/2	1	1	1	1
1.50	2	2	2	2	2	2	2	1/2	1	1	1	1
1.60	2	2	2	2	2	2	2	2	1/2	1	1	1

Table 4: Long-run increasing average consumption shares ($T = 500$, $\gamma_1 = 6$, $\gamma_2 = 8$)

The entries in the table indicate if the average consumption share of (an initially small) investor 1 and/or (an initially small) investor 2 have increased until $T = 500$ relative to the respective starting values of 0.1. '1', '2', and '1/2' indicate that investor 1's, 2's, or both investors' respective consumption share has increased on average. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is $\omega_0 \in \{0.1, 0.9\}$, the initial value of the state variable X is set to zero. The parameters of relative risk aversion are $\gamma_1 = 6$ and $\gamma_2 = 8$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.

$\omega_0 = 0.10$

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.33191	0.71874	0.90542	0.96478	0.98310	0.99097	0.99640	0.99740	0.99808	0.99853	0.99880	0.99901
0.50	0.12098	0.36582	0.67192	0.86689	0.94245	0.97237	0.99045	0.99380	0.99574	0.99682	0.99753	0.99808
0.60	0.05306	0.16473	0.38388	0.64691	0.83256	0.91735	0.97589	0.98557	0.99041	0.99336	0.99511	0.99631
0.70	0.02829	0.08290	0.20582	0.40892	0.63384	0.80481	0.94258	0.96674	0.97944	0.98639	0.99054	0.99303
0.80	0.01666	0.04882	0.11908	0.24419	0.43324	0.63115	0.87670	0.93023	0.95703	0.97213	0.98151	0.98692
0.90	0.01151	0.02914	0.07324	0.15211	0.27724	0.45353	0.76647	0.86025	0.91535	0.94626	0.96545	0.97655
1.10	0.00604	0.01409	0.03232	0.06762	0.12948	0.21986	0.48839	0.62753	0.75116	0.83687	0.89235	0.92848
1.20	0.00481	0.01109	0.02485	0.04882	0.09043	0.15954	0.37078	0.50668	0.64039	0.74735	0.82474	0.88357
1.30	0.00376	0.00869	0.01793	0.03694	0.06802	0.11817	0.28591	0.40482	0.52297	0.64502	0.74431	0.82177
1.40	0.00327	0.00694	0.01449	0.02876	0.05352	0.09117	0.22204	0.31636	0.42733	0.54408	0.65283	0.74587
1.50	0.00276	0.00611	0.01180	0.02204	0.04115	0.07033	0.17537	0.26035	0.34815	0.44785	0.55942	0.65986
1.60	0.00243	0.00503	0.00980	0.01809	0.03380	0.05793	0.14219	0.20620	0.28451	0.37260	0.48044	0.57879

$\omega_0 = 0.90$

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.92943	0.98847	0.99671	0.99866	0.99930	0.99959	0.99981	0.99986	0.99989	0.99991	0.99992	0.99993
0.50	0.63518	0.93387	0.98469	0.99500	0.99786	0.99887	0.99957	0.99970	0.99977	0.99983	0.99987	0.99989
0.60	0.30942	0.72612	0.93553	0.98196	0.99334	0.99695	0.99902	0.99937	0.99955	0.99968	0.99975	0.99980
0.70	0.16158	0.43617	0.78217	0.93721	0.97932	0.99163	0.99781	0.99862	0.99913	0.99939	0.99955	0.99966
0.80	0.09903	0.24786	0.53425	0.81734	0.94129	0.97777	0.99495	0.99715	0.99822	0.99883	0.99919	0.99942
0.90	0.06407	0.15662	0.33746	0.61719	0.84477	0.94227	0.98874	0.99400	0.99652	0.99776	0.99850	0.99899
1.10	0.03500	0.07739	0.15391	0.28924	0.50681	0.73292	0.94652	0.97433	0.98643	0.99220	0.99532	0.99691
1.20	0.02807	0.05845	0.11207	0.21094	0.36920	0.57105	0.89189	0.94821	0.97362	0.98563	0.99177	0.99458
1.30	0.02270	0.04683	0.08727	0.15762	0.26965	0.43271	0.80247	0.90302	0.95023	0.97371	0.98502	0.99075
1.40	0.01959	0.03685	0.07057	0.12499	0.20402	0.32933	0.68636	0.82305	0.91015	0.95274	0.97302	0.98435
1.50	0.01598	0.03082	0.05638	0.09749	0.16393	0.26055	0.55841	0.72414	0.84371	0.91648	0.95419	0.97389
1.60	0.01424	0.02618	0.04833	0.08060	0.12887	0.20651	0.45137	0.60840	0.75915	0.85930	0.92118	0.95635

Table 5: Long-term average consumption share of investor 1 ($T = 500$, $\gamma_1 = 4$, $\gamma_2 = 10$, i.i.d. consumption growth)

The table gives the average consumption share of investor 1 after 500 years. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is 10%, 50%, or 90%, consumption growth is i.i.d.. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$.

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	1	1	1	1	1	1	1	1	1	1	1	1
0.50	1/2	1	1	1	1	1	1	1	1	1	1	1
0.60	2	1/2	1	1	1	1	1	1	1	1	1	1
0.70	2	2	1/2	1	1	1	1	1	1	1	1	1
0.80	2	2	1/2	1/2	1	1	1	1	1	1	1	1
0.90	2	2	2	1/2	1/2	1	1	1	1	1	1	1
1.10	2	2	2	2	1/2	1/2	1	1	1	1	1	1
1.20	2	2	2	2	2	1/2	1/2	1	1	1	1	1
1.30	2	2	2	2	2	1/2	1/2	1	1	1	1	1
1.40	2	2	2	2	2	2	1/2	1/2	1	1	1	1
1.50	2	2	2	2	2	2	1/2	1/2	1/2	1	1	1
1.60	2	2	2	2	2	2	1/2	1/2	1/2	1/2	1	1

Table 6: Long-run increasing average consumption shares ($T = 500$, $\gamma_1 = 4$, $\gamma_2 = 10$, i.i.d. consumption growth)

The entries in the table indicate if the average consumption share of (an initially small) investor 1 and/or (an initially small) investor 2 have increased until $T = 500$ relative to the respective starting values of 0.1. '1', '2', and '1/2' indicate that investor 1's, 2's, or both investors' respective consumption share has increased on average. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is $\omega_0 \in \{0.1, 0.9\}$, the initial value of the state variable X is set to zero. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.10$. Consumption growth is i.i.d. with parameters $\mu_c = 0.02$ and $\sigma_c = 0.0252$.

$\omega_0 = 0.10$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.14766	0.56562	0.85689	0.94703	0.97576	0.98695	0.99476	0.99626	0.99721	0.99783	0.99825	0.99855
0.50	0.02924	0.15271	0.48689	0.78225	0.90838	0.95658	0.98575	0.99072	0.99357	0.99525	0.99635	0.99714
0.60	0.00917	0.04070	0.15720	0.43109	0.71192	0.86485	0.96200	0.97717	0.98507	0.98970	0.99250	0.99434
0.70	0.00400	0.01456	0.05152	0.16306	0.39547	0.65265	0.90298	0.94458	0.96605	0.97780	0.98470	0.98890
0.80	0.00214	0.00677	0.02073	0.06145	0.16763	0.36796	0.77449	0.87342	0.92454	0.95245	0.96874	0.97828
0.90	0.00131	0.00364	0.01006	0.02718	0.07207	0.17375	0.56324	0.73379	0.84101	0.90208	0.93775	0.95837
1.10	0.00064	0.00148	0.00346	0.00803	0.01843	0.04154	0.18421	0.32945	0.50615	0.66562	0.78062	0.85622
1.20	0.00049	0.00107	0.00235	0.00503	0.01078	0.02333	0.09939	0.19007	0.32717	0.48628	0.63547	0.75356
1.30	0.00039	0.00081	0.00165	0.00341	0.00693	0.01415	0.05622	0.10901	0.19530	0.32218	0.46985	0.61339
1.40	0.00032	0.00063	0.00124	0.00243	0.00476	0.00922	0.03408	0.06425	0.11744	0.20205	0.32097	0.45841
1.50	0.00027	0.00052	0.00097	0.00180	0.00338	0.00630	0.02181	0.04055	0.07225	0.12488	0.20730	0.31795
1.60	0.00024	0.00043	0.00078	0.00139	0.00254	0.00458	0.01471	0.02628	0.04635	0.07956	0.13430	0.21430
$\omega_0 = 0.90$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.92141	0.98635	0.99592	0.99828	0.99912	0.99947	0.99974	0.99981	0.99985	0.99987	0.99990	0.99991
0.50	0.59678	0.92329	0.98195	0.99387	0.99731	0.99855	0.99943	0.99960	0.99970	0.99976	0.99981	0.99984
0.60	0.23948	0.69062	0.92486	0.97831	0.99172	0.99610	0.99872	0.99916	0.99940	0.99956	0.99966	0.99973
0.70	0.09829	0.35753	0.74957	0.92724	0.97504	0.98962	0.99715	0.99821	0.99883	0.99918	0.99939	0.99953
0.80	0.04696	0.16823	0.46556	0.78791	0.92855	0.97269	0.99357	0.99630	0.99767	0.99845	0.99891	0.99920
0.90	0.02633	0.08556	0.25121	0.55292	0.81481	0.93042	0.98577	0.99229	0.99544	0.99707	0.99802	0.99861
1.10	0.01110	0.03031	0.07975	0.19633	0.41795	0.67769	0.93304	0.96733	0.98249	0.98980	0.99372	0.99584
1.20	0.00806	0.02029	0.04996	0.12001	0.26492	0.48816	0.86546	0.93465	0.96613	0.98117	0.98886	0.99279
1.30	0.00609	0.01450	0.03360	0.07683	0.16720	0.32941	0.75288	0.87568	0.93600	0.96546	0.98000	0.98764
1.40	0.00485	0.01066	0.02391	0.05229	0.10946	0.21958	0.60790	0.77777	0.88414	0.93804	0.96465	0.97899
1.50	0.00391	0.00830	0.01749	0.03661	0.07548	0.14946	0.45608	0.65077	0.80015	0.89075	0.93917	0.96453
1.60	0.00328	0.00664	0.01350	0.02692	0.05319	0.10403	0.33214	0.51155	0.68897	0.81772	0.89645	0.94077

Table 7: Long-term average consumption share of investor 1 ($T = 500$, $\gamma_1 = 6$, $\gamma_2 = 8$, i.i.d. consumption growth)

The table gives the average consumption share of investor 1 after 500 years. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is 10%, 50%, or 90%, the initial value of the state variable x is set to zero. The parameters of relative risk aversion are $\gamma_1 = 6$ and $\gamma_2 = 8$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.

$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	1	1	1	1	1	1	1	1	1	1	1	1
0.50	2	1	1	1	1	1	1	1	1	1	1	1
0.60	2	2	1	1	1	1	1	1	1	1	1	1
0.70	2	2	2	1	1	1	1	1	1	1	1	1
0.80	2	2	2	2	1	1	1	1	1	1	1	1
0.90	2	2	2	2	2	1	1	1	1	1	1	1
1.10	2	2	2	2	2	2	1	1	1	1	1	1
1.20	2	2	2	2	2	2	2	1	1	1	1	1
1.30	2	2	2	2	2	2	2	1/2	1	1	1	1
1.40	2	2	2	2	2	2	2	2	1/2	1	1	1
1.50	2	2	2	2	2	2	2	2	2	1/2	1	1
1.60	2	2	2	2	2	2	2	2	2	2	1/2	1

Table 8: Long-run increasing average consumption shares ($T = 500$, $\gamma_1 = 6$, $\gamma_2 = 8$, i.i.d. consumption growth)

The entries in the table indicate if the average consumption share of (an initially small) investor 1 and/or (an initially small) investor 2 have increased until $T = 500$ relative to the respective starting values of 0.1. '1', '2', and '1/2' indicate that investor 1's, 2's, or both investors' respective consumption share has increased on average. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is $\omega_0 \in \{0.1, 0.9\}$, the initial value of the state variable X is set to zero. The parameters of relative risk aversion are $\gamma_1 = 6$ and $\gamma_2 = 8$, both investors have a time preference rate of $\beta = 0.10$. Consumption growth is i.i.d. with parameters $\mu_c = 0.02$ and $\sigma_c = 0.0252$.

$\omega_0 = 0.10$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.13214	0.04585	0.02477	0.03857	0.02012	0.02099	0.05381	0.02520	0.10023	0.02578	0.07725	0.04056
0.50	0.17932	0.07067	0.08876	0.02562	0.03928	0.03758	0.07237	0.06653	0.03751	0.01895	0.08013	0.02196
0.60	0.20109	0.11323	0.09817	0.04945	0.09162	0.05473	0.05684	0.04172	-0.03577	0.06659	0.05277	0.06894
0.70	0.17410	0.14713	0.16534	0.12518	0.12728	0.00248	0.07291	0.02578	0.06404	0.03312	0.04436	0.04509
0.80	0.16711	0.21404	0.14565	0.17800	0.06753	0.07278	0.03965	0.02585	0.01508	0.10058	-0.02240	0.03497
0.90	0.15874	0.23013	0.12247	0.16973	0.09973	0.08047	0.05815	0.12233	0.10066	0.09165	0.01148	0.05307
1.10	0.17838	0.17289	0.18256	0.16298	0.17362	0.15193	0.11709	0.05427	0.05242	0.06641	0.02941	0.02355
1.20	0.22167	0.17901	0.21760	0.19100	0.19514	0.15268	0.15120	0.09456	0.09348	0.08605	0.10902	0.02872
1.30	0.22820	0.18713	0.23621	0.21211	0.25943	0.14629	0.18494	0.13806	0.11121	0.09217	0.10129	0.10290
1.40	0.17931	0.18697	0.23291	0.27108	0.19758	0.20383	0.16345	0.16262	0.13231	0.10036	0.09437	0.00438
1.50	0.18440	0.24261	0.15081	0.23970	0.22814	0.18205	0.17628	0.16983	0.15471	0.15294	0.05073	0.11491
1.60	0.14672	0.21658	0.19991	0.20623	0.23892	0.26498	0.15711	0.19215	0.14494	0.15072	0.10800	0.08476
$\omega_0 = 0.90$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.10197	0.07146	0.06364	0.04036	0.05371	-0.00341	0.07749	-0.00641	-0.00809	0.00748	0.00942	0.02470
0.50	0.10079	0.08758	0.00569	0.07521	0.05578	0.10521	0.09382	0.03402	0.04896	0.02482	0.06221	0.04113
0.60	0.14232	0.10706	0.09078	0.04287	0.08584	-0.03824	0.04499	0.04402	0.08248	0.07471	0.09405	-0.02633
0.70	0.19076	0.17114	0.12765	0.10905	0.05998	0.07936	0.08190	0.04400	0.05342	0.07526	0.01968	0.02324
0.80	0.22150	0.19980	0.11655	0.12151	0.10501	0.07941	0.06393	0.07395	0.01312	0.03998	0.03518	0.02726
0.90	0.26444	0.28320	0.15262	0.07580	0.09636	0.04340	0.03306	0.08252	0.03191	0.01607	0.02936	0.13611
1.10	0.23357	0.26757	0.20798	0.18648	0.12505	0.17159	-0.00982	0.06628	0.14132	0.05841	0.03464	0.03569
1.20	0.23908	0.26573	0.23863	0.23328	0.19282	0.05496	0.09720	0.08350	0.07349	0.05079	0.05535	0.08788
1.30	0.22273	0.23434	0.26860	0.27235	0.21932	0.17919	0.15305	0.06205	0.02640	0.08386	0.11661	0.03208
1.40	0.17951	0.24911	0.23330	0.25426	0.23073	0.21202	0.10238	0.10215	0.08762	0.07947	0.08405	0.08545
1.50	0.25469	0.25769	0.24601	0.22779	0.25665	0.20287	0.09052	0.14098	0.06601	0.10550	0.04817	0.09341
1.60	0.26233	0.28899	0.27380	0.25988	0.24730	0.26709	0.19133	0.16750	0.11964	0.06709	0.08031	0.06643

Table 9: Correlation of X and ω ($T = 500$, $\gamma_1 = 4$, $\gamma_2 = 10$)

The table gives the correlation of x and the consumption share of investor 1 after 500 years. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is 10%, 50%, or 90%, the initial value of the state variable x is set to zero. The parameters of relative risk aversion are $\gamma_1 = 4$ and $\gamma_2 = 10$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.

$\omega_0 = 0.10$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.08127	0.07358	0.00133	0.05092	-0.02411	-0.02420	0.00755	0.01360	-0.04566	0.01542	-0.01260	-0.03868
0.50	0.07476	0.13635	0.02922	0.06851	0.05329	0.06113	-0.01875	0.02021	-0.03153	-0.03180	-0.00664	0.03594
0.60	0.15062	0.04995	0.06398	0.03772	0.04298	-0.00131	-0.01739	-0.03703	-0.03733	0.01315	0.01728	-0.05321
0.70	0.15720	0.08371	0.06801	0.09557	0.07883	0.01662	0.04579	0.04430	0.02071	0.03782	0.02356	-0.00627
0.80	0.12050	0.17995	0.12311	0.15272	0.03333	0.08119	0.04945	0.05014	0.03587	0.09813	-0.03816	0.03165
0.90	0.08428	0.13443	0.20140	0.10188	0.09859	0.02992	0.05806	0.06049	0.06420	-0.00330	0.01878	-0.02553
1.10	0.22254	0.10966	0.18295	0.13877	0.12831	0.09031	0.09523	0.00412	0.04457	0.03035	0.04106	0.00034
1.20	0.17952	0.14453	0.13563	0.20906	0.10861	0.12496	0.08016	0.04267	0.04020	0.07571	0.06680	0.01514
1.30	0.17848	0.24759	0.18197	0.14651	0.12178	0.11583	0.06565	0.12709	0.08914	0.12700	0.00864	0.07343
1.40	0.21942	0.23919	0.17364	0.19957	0.15013	0.13892	0.03358	0.07682	0.15873	0.06285	0.06934	0.01256
1.50	0.24126	0.20055	0.17031	0.19357	0.11076	0.14520	0.16811	0.07838	0.12899	0.08368	0.04082	0.05810
1.60	0.21684	0.16746	0.16054	0.18782	0.18295	0.12386	0.19405	0.09966	0.15139	0.14245	0.12669	0.13557
$\omega_0 = 0.90$												
$\downarrow \psi_2 \mid \psi_1 \rightarrow$	0.40	0.50	0.60	0.70	0.80	0.90	1.10	1.20	1.30	1.40	1.50	1.60
0.40	0.08360	0.00997	0.00050	0.01157	-0.01714	0.03433	0.02156	-0.00187	0.02248	-0.01046	-0.02435	-0.03364
0.50	0.14133	0.09565	0.03667	0.07768	-0.00721	0.01855	0.01096	0.06624	-0.02004	-0.00589	-0.04163	0.01861
0.60	0.19628	0.12459	0.09758	0.09205	0.07904	-0.01212	0.01394	0.01385	0.00382	-0.00754	0.07001	-0.03007
0.70	0.24841	0.18854	0.13190	0.08597	-0.03255	0.05880	0.04550	-0.01784	0.00618	0.03545	-0.09420	0.02972
0.80	0.24388	0.24702	0.21480	0.08528	0.09056	0.05551	0.05376	0.03417	0.03310	0.00467	0.00724	0.00973
0.90	0.28184	0.23163	0.22652	0.14128	0.11072	0.07786	0.03083	0.01895	0.07084	0.04257	0.02747	0.00064
1.10	0.27407	0.23313	0.24452	0.30184	0.14569	0.13042	0.10250	0.07097	0.07603	0.08412	0.07679	-0.01356
1.20	0.25932	0.29096	0.21359	0.27402	0.21768	0.17784	0.13484	0.07554	0.07973	0.06187	-0.04501	0.05471
1.30	0.21367	0.29629	0.27633	0.22331	0.19087	0.24143	0.14452	0.11367	0.11817	0.03713	0.06989	0.04218
1.40	0.23946	0.29489	0.28127	0.26666	0.23529	0.20245	0.18565	0.10923	0.12982	0.07401	0.06326	0.05112
1.50	0.25743	0.25640	0.26164	0.24348	0.26998	0.24208	0.13501	0.12255	0.09013	0.04324	0.12467	0.12118
1.60	0.26668	0.29672	0.27851	0.29225	0.20866	0.28904	0.21956	0.17498	0.16742	0.07011	0.10426	0.12483

Table 10: Correlation of X and ω ($T = 500$, $\gamma_1 = 6$, $\gamma_2 = 8$)

The table gives the correlation of x and the consumption share of investor 1 after 500 years. The calculation is based on a Monte Carlo simulation with 1,000 runs. The initial consumption share is 10%, 50%, or 90%, the initial value of the state variable x is set to zero. The parameters of relative risk aversion are $\gamma_1 = 6$ and $\gamma_2 = 8$, both investors have a time preference rate of $\beta = 0.10$. The drift and volatility of consumption are $\mu_c = 0.02$ and $\sigma_c = 0.0252$. The parameters for the long-run growth rate process X are $\kappa_x = 0.3$ and $\sigma_x = 0.0114$.