Benchmarking the Performance of Funding Risk Measures for Credit Facilities

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Abstract

The newly-gained attention towards liquidity risk led to increased demand for a sound risk identification of contingent products like credit facilities. In order to reliably assess the liquidity risk of credit facilities on a portfolio level, the eligibility of the applied portfolio measure must be well-founded. However, to quantitatively evaluate the performance of a specific measure, the (potentially unknown) future portfolio draw-down distribution is needed. We address this problem by constructing an environment for the generation of data sets for varying assumptions. Data histories are simulated for a wide parameter range, allowing us to study the performances of different portfolio measures currently discussed in literature, for multiple scenarios. Via a criteria-based analysis we find that a Monte-Carlo simulation approach built around the modeling of risk factors which incorporates an additional portfolio correlation structure, provides the best results for the estimation of a future draw-down distribution.

Key words: Credit Facility, Line of Credit, Loan Commitment, Liquidity Risk, Funding Risk, Portfolio Model

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1 Introduction

The financial meltdown of recent years clearly highlighted the importance of liquidity for a proper functioning of financial markets.[1] As a consequence, regulators have defined quantitative measures (Pillar 1 of Basel) and strengthened qualitative requirements (Pillar 2 of Basel) for the management of liquidity risk.[2] In particular, it is now more and more required to use reliable *quantitative* models to manage certain risks including elements of funding risk.[3] In this context, a portfolio model for credit and liquidity facilities[4] serves as a considerable contribution for a robust internal liquidity risk management framework.

In line with regulatory requirements, we distinguish two components as relevant output of a quantitative portfolio model for liquidity and credit facilities. First, the expected volume of future draw-downs of facilities[5] should be considered as part of a business as usual cash flow forecast.[6] Furthermore, financial institutions should consider potential deviations from expected future draw-downs, i.e., contingent future draw-downs, in their management framework (in particular as a component of their liquidity stress tests).

Current literature on quantitative portfolio models regarding credit and liquidity facilities is sparse. Numerous approaches for the assessment of credit and market portfolio risk have been proposed.[7] However, only very few models concentrate on liquidity risk, in particular if associated with credit facilities. While expected future draw-downs are generally

[1] See BIS (2008), p.12.

[2] Usually, liquidity risk is further differentiated into funding liquidity risk and market liquidity risk. "Funding liquidity risk is the risk that the firm will not be able to meet efficiently both expected and unexpected current and future cash flow and collateral needs without affecting either daily operations or the financial condition of the firm. Market liquidity risk is the risk that a firm cannot easily offset or eliminate a position at the market price because of inadequate market depth or market disruption." (BIS (2008)).

[3] BIS (2008) states that "given the customized nature of many of the contracts that underly undrawn commitments and off-balance sheet instruments, triggering events for these contingent liquidity risks can be difficult to model." However, BIS (2008) requires a financial institution "to implement systems and tools to analyze these liquidity trigger events effectively".

[4] For the purpose of this paper, we describe liquidity and credit facilities on the basis of BIS (2010), which differentiates facilities according to the purpose a customer holds such a line. Back-up facilities, put in place purely for the purpose of refinancing debt in tense situations, are defined as liquidity facilities. General working capital facilities (e.g. revolving credit facilities), put in place for general and/or working capital purposes, are defined as credit facilities.

[5] In the following, we will always refer to *expected future draw-downs.*

[6] See BIS (2008), p. 12.

[7] See, e.g. Grundke (2005) and Jorion (2006).

calculated as some sort of mean volume of (historical) draw-downs,[8] different approaches for the assessment of contingent draw-downs have been proposed.[9]

A complete survey, discussing strengths and weaknesses of the individual quantitative approaches has so far not been conducted. The extent to which current regulatory requirements are covered is also unclear. A comparison of different model's performances regarding the 'correct' assessment of portfolio liquidity risk is limited by the necessity to confirm estimated liquidity needs from actual data. The models' predictions for future portfolio draw-downs (as well as predictions for risk quantiles) may well be compared with actual realizations of these properties (if available), but even then offer only a *indication* for the measures' adequacy. A sound model should provide a methodologically well-grounded estimate, not only for expected future draw-downs, but more or less for the complete distributional form of future draw-downs. This is of importance, as the calculations of contingent future draw-downs generally depend on specific characteristics of the underlying portfolio distribution. If contingent draw-downs are defined on the basis of the by now ubiquitous Value-at-Risk concept, confirmation of a model's capability to capture, e.g., a 95% quantile, will rely on a long history of available data (which can therefore not be used to calibrate the model in the first place). We attempt to overcome these issues by generating fictional data sets on the basis of few axiomatical assumptions. This setup will allow us to study the performance of a funding risk measure with regard to an explicitly simulated portfolio draw-down distribution.

Further, it is necessary to devise a catalogue of criteria according to which different funding risk measures may be compared. For instance, a reasonably 'safe' measure in terms of covered downside risk (highly desirable) may easily be set up by calibrating (almost) any given portfolio measure on the premise of extreme environmental conditions. This may in turn demand large funding needs (highly undesirable).

The content of the paper is organized as follows: In Section 2 we discuss possible approaches for the calculation of expected and contingent future draw-downs. Desirable criteria, a 'good' measure should meet are derived in Section 3. A flexible setup for the simulation of draw-down histories for a fictitious portfolio is established in Section 4. All introduced portfolio measures are afterwards evaluated on the basis of data histories that are generated for eight different combinations of environmental assumptions (Section 5). Section 6 sums up with some concluding remarks.

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8 See Matz and Neu (2007) and Heidorn *et al.* (2008).

9 A detailed overview on current theoretical literature is provided in Section 2.

2 Funding risk measures

In this section, we discuss different approaches for the estimation of expected and contingent future draw-downs. Let T denote the number of observation dates. We assume the existence of a draw-down 'history' $(d_{i,t})_{1\leq i\leq n, 1\leq t\leq T}$, where i refers to the specific credit line and t denotes the time of the draw-down. For the sake of simplicity, we will assume that the number of customers n is fixed and each customer has a credit line with a granted volume[10] of 1 € at her disposal.[11]

In the following, we discuss *three* different approaches for the estimation of expected and contingent future draw-downs, all of which are derived from theoretical literature. Each model provides at least one value ν to estimate portfolio liquidity risk and directly addresses expected and contingent future draw-downs by means of a decomposition

$$\nu = EDD_\nu + CDD_\nu. \tag{1}$$

2.1 Naive Approach

A **naive approach**, which is commonly known from quantitative 'Bodensatz'-Models for liquidity and interest rate risk,[12] consists of a simple utilization of the historical portfolio volatility for the assessment of contingent draw-downs. We apply the standard unbiased estimators for portfolio mean and standard deviation:[13]

$$\hat{\mu}_{PF} = \frac{1}{T}\sum_{t=1}^{T} d_t^{PF} \tag{2}$$

$$\hat{\sigma}_{PF} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(d_t^{PF} - \hat{\mu}_{PF})^2} \tag{3}$$

10 The granted volume denotes the €-amount granted by a financial institution to a customer as the volume which can be drawn.

11 This assumption poses no constraint for the models which are about to be introduced. An adjustment to non-constant granted amounts is straight forward in each case.

12 See Bartetzky *et al.* (2008), p. 182 – 187.

13 Where $d_t^{PF} = \sum_{i=1}^{n} d_{i,t}$.

A measure for funding risk is then given through

$$\nu_{naive} = \hat{\mu}_{PF} + \alpha \cdot \hat{\sigma}_{PF}, \tag{4}$$

where the parameter $\alpha > 0$ can be determined according to a financial institution's risk tolerance. In this setting, $\hat{\mu}_{PF}$ corresponds to expected future draw-downs ($EDD_{\nu_{naive}}$), where $\hat{\mu}_{PF}$ can be interpreted as the sum of all mean line utilizations μ_i.[14] Contingent future draw-downs ($CDD_{\nu_{naive}}$) are addressed by $\alpha \cdot \hat{\sigma}_{PF}$. For the purpose of this paper, we estimate $CDD_{\nu_{naive}}$ with $\alpha = 1$.

2.2 Model of Heidorn *et al.* (2008)

Heidorn *et al.* (2008) analyze how to set up and calibrate models for specific portfolios of facilities (i.e., specific sub-portfolios of term facilities and rolling credit facilities) in the context of an industry project. Their data set suffers from typical constraints of data and their availability.[15] In accordance with standard Value-at-Risk measures broadly employed in credit risk models, **Heidorn *et al.* (2008)** calculate a funding quantile (for a risk level of $\beta = 5\%$) for the portfolio in the following way:

For each line of credit $i \in \{1, ..., n\}$, the mean utilization $\mu_i = \frac{1}{T}\sum_{t=1}^{T} d_{i,t}$ is calculated. Additionally, the authors determine the 95% quantile $q_{95\%}^{Heidorn}$ from the set of all observable deviations $\{d_{i,t} - \mu_i | 1 \leq i \leq n, 1 \leq t \leq T\}$.

The measure used for total funding risk is given through

$$\nu_{Heidorn} = \hat{\mu}_{PF} + n \cdot q_{95\%}^{Heidorn}. \tag{5}$$

Again, $\hat{\mu}_{PF}$ corresponds to expected future draw-downs ($EDD_{\nu_{Heidorn}}$), whereas contingent future draw-downs are addressed by $n \cdot q_{95\%}^{Heidorn}$ ($CDD_{\nu_{Heidorn}}$).

As Heidorn *et al.* (2008) themselves state, the resulting portfolio quantile is extremely conservative and consequently omits diversification effects of draw-downs between single facilities. An event in which the contingent future draw-down exceeds the quantile $q_{95\%}^{Heidorn}$ corresponds to an exceeding of *every* line's individual 95% risk quantile (on average). If

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14 To follow this, we simply calculate $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \left(\frac{1}{T} \sum_{t=1}^{T} d_{i,t} \right) = \sum_{t=1}^{T} \left(\frac{1}{T} \sum_{i=1}^{n} d_{i,t} \right) = \hat{\mu}_{PF}$, where $\mu_i = \frac{1}{T} \sum_{t=1}^{T} d_{i,t}$.

15 Specifically, most data histories in the applied data set are incomplete.

credit line draw-downs are not perfectly correlated, this event should turn out to occur extremely infrequent. Heidorn *et al.* (2008) argue that the proposed model could be enhanced theoretically, but practical experience shows that the general model setup is in any case driven by the quality of available data.

2.3 Extension of the model of Duffy *et al.* (2005)

Duffy *et al.* (2005) derive a quantitative portfolio model for the measurement of funding risk on a portfolio level. A customer's draw-down is derived on the basis of a simulated credit rating following a mark-to-market (mtm) credit portfolio model logic. Additionally, an expert-driven system corresponding to the usage of term-out options is employed.[16] However, neither are the used model parameters analyzed in detail nor is a dependency structure regarding the draw-downs of credit lines, contingent on the rating, considered. Hence, we will use Duffy *et al.* (2005) as a basis to set up an enhanced model framework.[17]

In line with Duffy *et al.* (2005), the customers' *credit rating* is included as a driver of individual future draw-downs. Future rating states $r_{i,T+1}$ for date $T+1$ are simulated for each customer $i \in \{1, ..., n\}$, based on the current rating $r_{i,T}$. Let $[a]$ and $[b]$ denote two rating classes.[18] The migration probability $\hat{p}_{[a]\to[b]}$ is estimated from the observed data via

$$\hat{p}_{[a]\to[b]} = \frac{|\{(i,t)|1 \le i \le n, 1 \le t \le T-1, r_{i,t} = [a], r_{i,t+1} = [b]\}|}{|\{(i,t)|1 \le i \le n, 1 \le t \le T-1, r_{i,t} = [a]\}|}. \tag{6}$$

The estimation for $\hat{p}_{[b]\to[a]}$ follows analogously. The probabilities to remain in each state are then given through $\hat{p}_{[a]\to[a]} = 1 - \hat{p}_{[a]\to[b]}$ and $\hat{p}_{[b]\to[b]} = 1 - \hat{p}_{[b]\to[a]}$.

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16 In analogy to Duffy *et al.* (2005), Paulsen (2007) bases the draw-downs of credit lines on the realization of a risk factor (the credit spread or the rating). While Duffy *et al.* (2005) allow for flexible draw-downs contingent on the realization of the risk factor, Paulsen (2007) assumes either full or no drawing per credit line (based on the realization of the considered risk factor). Additionally, he analyzes the impact of some input parameters on the portfolio distribution. As his approach is more or less a generalization of Duffy *et al.* (2005), we will restrict our analysis to the model of Duffy *et al.* (2005).

17 The work of Duffy *et al.* (2005) was established in the context of an industry project and not all features of their model are transparently detailed in the paper. Consequently, we have to extrapolate some features of the model.

18 We restrict our description to a world, where only two rating states are allowed. Rating $[a]$ corresponds to a 'good' rating quality, whereas rating $[b]$ indicates a reduced credit worthiness of the corresponding customer. However, the expansion to an approach with more states is straight forward.

In order to obtain a portfolio draw-down for $T + 1$, the rating class each customer finds herself in at $T + 1$ is simulated first, based on the actual state at T and the estimated transition probabilities. Ratings are determined independently for all customers. This simple approach does not capture dependencies between different customers on the level of the rating process. Nevertheless, the information about a customer's rating is in the following utilized for the estimation of a portfolio draw-down distribution. Of course, the simulation of rating processes for actual data may be implemented on the basis of a rating factor model. Rather than implementing such a full scale factor model into an estimation model, we will analyze whether an (albeit rough) estimate for customer ratings generally improves forecasting performance.

Given a specific rating realization $[a]$ or $[b]$, customers *draw from the corresponding marginal distributions* for individual draw-downs $F_{[a]}$ or $F_{[b]}$, where $[a]$ or $[b]$ denotes the respective rating state. The approximation of the true marginal distributions $F_{[a]}$ and $F_{[b]}$ is straightforward. Individual portfolio draw-downs are simulated from the sets of historical utilizations $V_{[a]}$ and $V_{[b]}$:

$$V_{[a]} = \{d_{i,t} | 1 \leq i \leq n, 1 \leq t \leq T, r_{i,t} = [a]\}. \tag{7}$$

The definition of $V_{[b]}$ again occurs analogously. The resulting empirical distributions serve as estimates $\hat{F}_{[a]}$ and $\hat{F}_{[b]}$.[19]

Considering that correlation effects have thus far not been covered through the rating simulation, we specifically emphasize *the estimation of a dependency structure at a draw-down level.*[20] The estimation approach is set up in the following way:

- We do not estimate historical correlations on the basis of absolute draw-downs $(d_{i,t})_{i,t}$. Instead, we first 'correct' these draw-downs according to the respective estimated marginal distribution $\hat{F}_{r(i,t)}$ through:[21]

$$\hat{u}_{i,t} = \min \left\{ \hat{F}^{-1}_{r_{i,t}} \left(d_{i,t} \right) \right\}. \tag{8}$$

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19 The historical draw-down data could also be used to calibrate specific classes of marginal distributions, e.g. normal or gamma distributions. Such a process could speed up technical implementation of a simulation approach.

20 At this stage of the model, we explicitly enhance the model of Duffy *et al.* (2005).

21 As $\hat{F}_{r(i,t)}$ is a discrete distribution, $\hat{F}^{-1}_{r(i,t)} \left(d_{i,t} \right)$ may not be unique. Hence, we apply the minimum operator.

This leaves us with n observed 'relative' draw-down histories $\hat{u}_1, ..., \hat{u}_n \in [0,1]^T$, where $\hat{u}_i = (\hat{u}_{i,1}, ..., \hat{u}_{i,T})$.

- The correlation structure is modeled with a Gaussian copula. A simulation via this approach necessarily demands a positive (semi-)definite, symmetric correlation matrix Γ as input parameter. A (first) estimate for Γ is retrieved from the matrix of (empirical) correlations between relative draw-down histories, $(\rho(\hat{u}_i, \hat{u}_j))_{i,j}$.[22] While this matrix is symmetric by definition of $\rho(\cdot,\cdot)$, positive (semi-)definiteness is generally not assured. We apply a quadratically convergent Newton method to compute a 'similar' positive definite correlation matrix $\hat{\Gamma}$, according to a pre-specified minimal eigenvector.[23] The 'similarity'[24] of the resulting estimate $\hat{\Gamma}$ depends on the shape of the original correlation matrix Γ, the amount of reshaping that is necessary to transform $(\rho(\hat{u}_i, \hat{u}_j))_{i,j}$ into a valid correlation matrix and (most importantly) the length of the data history. Fully accepting the shortcomings connected with a less than perfect estimation of Γ, we nonetheless simulate the portfolio's correlation structure based on the estimate $\hat{\Gamma}$ to analyze the quality of the forecast while including an estimated correlation structure.[25]

- As a final step, we simulate a random vector $\widetilde{X} = (\widetilde{X}_1, ..., \widetilde{X}_n) \sim \mathcal{N}(0, \hat{\Gamma})$ according to a Gaussian copula with correlation matrix $\hat{\Gamma}$. For a vector of realizations $(x_1, ..., x_n)$, each customer's (absolute) draw-down is calculated from the respective marginal distribution at $t = T$:

$$\hat{d}_{i,T+1} = \hat{F}_{r(i,T)}(\Phi(x_i)). \tag{9}$$

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22 $\rho(\cdot)$ denotes Pearson's linear (empirical) correlation coefficient. As $\rho(\cdot)$ is ill defined for constant (relative) data histories, we need to exclude these data histories for all customers.

23 Let $A = (a_{ij}), B = (b_{ij})$ two matrices with identical size. The distance between A and B is measured via the Frobenius norm: $\|A - B\|_{\mathrm{F}} = \sqrt{\sum_{i,j} |a_{ij} - b_{ij}|^2}$. See Qi and Sun (2006) for a theoretical background for the applied Newton algorithm, as well as a numerical illustration.

24 See Footnote 23.

25 In Section 4 we introduce a mechanism for the construction of data histories $(d_{i,t})_{1 \le i \le n, 1 \le t \le T}$, which exhibit certain correlation structures. For a control simulation we generated various data histories with a very long time horizon T. For $\hat{u}_{i,t}$ as in (8), we then employ $(\rho(\hat{u}_i, \hat{u}_j))_{i,j}$ as a proxy for the 'true' correlation structure of such a very extensive data history. Calculation of the correlation structure for the same data history, in which only a smaller time horizon $T' < T$ is considered, shows large deviations from the 'actual' correlation entry in $(\rho(\hat{u}_i, \hat{u}_j))_{i,j}$. This seems intuitive, considering that short histories for two uncorrelated customers have a higher 'chance' of similar (or anti-similar) movements. However, results indicate that empirical correlations for each truncated history disperse around their 'true' value. Also, large (absolute) correlations are preserved somewhat better in such short data histories, due to the fact that the fluctuation range for these correlations is more limited.

Simulation of (i) future credit ratings and (ii) individual draw-downs for each customer results in a draw-down vector $(\hat{d}_{1,T+1}, ..., \hat{d}_{n,T+1})$. The (simulated) future portfolio draw-down is then given through

$$\hat{d}^{PF}_{T+1} = \sum_{i=1}^{n} \hat{d}_{i,T+1}. \tag{10}$$

The simulation of m realizations for $\hat{d}^{PF}_{T+1}$ in a Monte-Carlo setting yields an estimate $\hat{PF}$ for the 'true' portfolio draw-down distribution $\widetilde{PF}$. From $\hat{PF}$, standard portfolio risk measures such as quantiles and variance-based ratios can be derived. We calculate

$$\nu_{\hat{\sigma}} = \hat{\mu}_{\hat{PF}} + \alpha \cdot \hat{\sigma}_{\hat{PF}} \tag{11}$$

and

$$\nu_{95\%} = \hat{\mu}_{\hat{PF}} + q^{95\%}_{\hat{PF}}, \tag{12}$$

where $\hat{\mu}_{\hat{PF}}$, $\hat{\sigma}_{\hat{PF}}$ and the 95% quantile $q^{95\%}_{\hat{PF}}$ are estimated from the simulated portfolio distribution $\hat{PF}$ in analogy to (2) and (3).[26] The parameter α can be chosen to reflect a portfolio manager's risk tolerance (compare the setup of ν_{naive} via equation (4)). Again, we employ $\alpha = 1$ for the calculation of $CDD_{\nu_{\hat{\sigma}}}$. In both cases $\hat{\mu}_{\hat{PF}}$ refers to expected future draw-downs ($EDD_{\nu_{\hat{\sigma}}} = EDD_{\nu_{95\%}}$). Contingent future draw-downs are addressed by $\alpha \cdot \hat{\sigma}_{\hat{PF}}$ ($= CDD_{\nu_{\hat{\sigma}}}$) and $q^{95\%}_{\hat{PF}}$ ($= CDD_{\nu_{95\%}}$). In both approaches, expected and contingent future draw-downs are not directly derived from historical data, as opposed to the approaches introduced in (4) and (5). Historical information is therefore used exclusively to estimate the model parameters $\hat{p}_{[a]\rightarrow[b]}, \hat{p}_{[b]\rightarrow[a]}$, $\hat{\Gamma}$, $F_{[a]}$ and $F_{[b]}$, whereas future draw-downs are simulated on the current state of the economy (i.e., the states all portfolio customers find themselves in).

[26] Both measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ are calculated from the Monte-Carlo results for $\hat{PF}$.

3 Criteria for a funding risk measure's performance

In the following we derive relevant criteria, according to which the quality regarding the assessment of future portfolio funding needs can be compared for all introduced measures. For the definition of criteria, we assume the existence of a sufficiently large number of draw-down histories, which will be simulated in Section 4. To allow for varying environmental circumstances, we develop different types of data generating processes.[27] For each specific environment S, a set of M draw-down histories $H_1^S, ..., H_M^S$ is generated, where $H_m^S \in \mathbb{R}^{n\times(T+1)}$ for all m. Each 'history' therefore includes realizations for future draw-downs at $T+1$. The resulting simulated portfolio draw-down distribution at $T+1$ is denoted by $\widetilde{PF}_S$.[28] Expected values, standard deviations and all other properties regarding $\widetilde{PF}_S$ are calculated with respect to all simulated histories for the underlying environment S.[29]

1. The definition of risk measures in Section 2 allows us to individually address risks arising from expected as well as contingent future draw-downs. Generally, estimated contingent draw-downs may be adjusted to better reflect financial institutions' risk tolerances. According to the construction via $\nu = EDD + CDD$, risk neutral financial institutions are interested in covering EDD as precisely as possible, in order not to fund any excess liquidity risk. Consequently expected draw-downs can be regarded as a sort of anchor point for further analysis and the derivation of contingent draw-downs. Furthermore, regulators require an appropriate consideration of EDD as part of a business as usual cash flow forecast for financial institutions.[30] Therefore, the more precise expected future draw-downs are estimated, the better risk tolerances can be accounted for (based on the residual contingent draw-down $CDD = \nu - EDD$). Furthermore, regulatory requirements are considered precisely in the business as usual cash flow forecast. To analyze the extent to which an **exact determination of expected future draw-downs** is accomplished, we calculate the expected deviation between estimated and realized portfolio

27 More precisely, we calculate data histories for eight combinations of different environmental assumptions, which will be outlined in Sections 4.3-4.5.

28 Each history H_m^S provides a single value for the portfolio draw-down at $T+1$. $\widetilde{PF}_S$ therefore is a discrete distribution composed of M data points.

29 E.g., the mean liquidity forecast for environment S via the naive approach outlaid in Section 2.1 is calculated through $\mathbb{E}_S(\nu_{naive}) = \frac{1}{M}\sum_{m=1}^{M}\nu_{naive}(H_m^S)$, where $\mathbb{E}_S$ denotes the expectation operator with regard to the environment S. Note that the draw-downs for time $T+1$ (i.e., n data points stored in each history H_m^S) are not used to calculate (or *simulate*, in the cases of $\nu_{\tilde{\sigma}}$ and $\nu_{95\%}$) future portfolio funding needs.

30 See BIS (2008), p. 12.

draw-downs for each measure. For each environment S and each portfolio measure $\nu \in \{\nu_{naive}, \nu_{Heidorn}, \nu_{\hat{\sigma}}, \nu_{95\%}\}$ we calculate $\mathbb{E}_S(EDD_\nu - \widetilde{PF}_S)$, which should be close to zero if ν on average captures expected future draw-downs correctly. A possible structural misestimation for expected future draw-downs is thus identified. To account for absolute deviations, we also calculate $\mathbb{E}_S(|EDD_\nu - \widetilde{PF}_S|)$.[31]

2. One major function of a funding risk measure ν is to reliably capture downside funding risk. If a precise probability can be assigned for the events in which ν falls short of $\widetilde{PF}_S$, financial institutions may align their risk tolerances accordingly. This does not mean, that a measure ν which underestimates some realizations of $\widetilde{PF}_S$ for any environment S is necessarily a useless measure. Rather, the amount of total downside risk covered by ν should be known precisely. Again, financial institutions are also required to consider contingent draw-downs in line with their risk tolerances as part of liquidity risk stress testing.[32] Accordingly, we calculate the **empirical shortfall probabilities** $\mathbb{P}_S(\nu - \widetilde{PF}_S < 0)$ for all environments S, to analyze, in how far a specific shortfall probability can be assigned to the portfolio measure ν. A simulated data history H_m^S will be identified as a shortfall for ν, if $\nu(H_m^S) - \widetilde{PF}_S(H_m^S) < 0$.[33]

3. A major difference in the definition of risk measures in Section 2 lies in the **processing of relevant information**. Although a strong focus on recent information in the definitions of $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ seems rational from our point of view, the total impact of such concentration on specific information is ex ante not clear. The 'historical' measures ν_{naive} and $\nu_{Heidorn}$ do not distinguish between older and more recent information. Such historical measures do not reflect portfolio risk on actual (possibly unusual) conditions, but rather assume a medium scenario, especially due to the fact that underlying risk factors, driving the portfolio measures, are not explicitly considered. Hence, we expect the inclusion of additional recent information (i.e., information about a customer's rating state $[a]$ or $[b]$) to provide a more *stable* estimate. This means that historical portfolio measures may on average forecast liquidity needs as good as measures distinguishing between older and more recent

31 As a backup check, we also calculate $\mathbb{E}_S[(EDD_\nu - \widetilde{PF}_S)^2]$. Results indicate no deviations from the conclusions drawn from the calculation of $\mathbb{E}_S(EDD_\nu - \widetilde{PF}_S)$ and are therefore not documented in Section 5.

32 See BaFin (2010), BTR 3.1. and BTR 3.2 and for the latest consultation paper BaFin (2012).

33 $\widetilde{PF}_S(H_m^S)$ denotes the portfolio draw-down at $T+1$ for the data history (H_m^S).

information.[34] However, the forecasting quality may fluctuate more severely around the 'correct' liquidity needs. To analyze this thesis, we first calculate the figure

$$\sqrt{\mathbb{V}_S(\nu - \tilde{PF}_S)} := \sqrt{\frac{1}{M-1}\sum_{m=1}^{M}(\nu(H_m^S) - \tilde{PF}(H_m^S))^2} \tag{13}$$

for all environments S, in order to account for the estimate's volatility.[35] The more precise relevant information concerning a customer's drawing behavior is utilized, the less volatile this estimate should turn out.

4. Another important property which has not been explicitly covered by our model setup is **reliance on provided data**. To compare the introduced measures in a sound way, all information for the calculation of the respective measures is assumed available. However, a real data set may contain incomplete data histories, missing values or even whole absent statistics needed for a measure's calibration. In risk management practice, data availability, consistency and quality may pose bottlenecks for sound models. In this context, we will discuss
   - the quantity and quality of data requirements for the respective measure;
   - the sensitivity of a measure towards poor or disturbed data quality;
   - possibilities to compensate poor data quality for the respective measure, e.g. through the utilization of additional external data/information concerning the model parameters.

[34] This characteristic is addressed by criterion 1.

[35] For the calculation of the variance, we apply the standard unbiased estimator in analogy to (3): $\mathbb{V}(\nu - \tilde{PF}_S) = \frac{1}{M-1}\sum_{m=1}^{M}(\nu(H_m^S) - \tilde{PF}(H_m^S))^2$.

4 Data history generation

The application of the criteria developed in the last section demands a variety of data histories. Moreover, in order to evaluate the performance of the funding risk measures derived in Section 2 in any quantitative form, 'real' data from an actual portfolio does not suffice for the following reason:

Suppose a draw-down history $(d_{i,t})_{1\leq i\leq n,1\leq t\leq T}$ exists for the underlying Portfolio. We could of course proceed and estimate a portfolio distribution $\hat{PF_T}$, i.e., the portfolio draw-down at $t = T$, estimated on the basis of the information available at $t = T - 1$. The resulting distribution $\hat{PF_T}$ can then only be compared to the draw-downs $(d_{i,T})_{1\leq i\leq n}$, i.e. to a *single* point in time, which does not give any reliable indication about the estimate's quality.

4.1 Simulation setup

We thus set up an environment for the simulation of artificial data histories for a fictitious portfolio.[36] This may seem somewhat contradictory to our explained goal of assessing the quality of various risk measures. If the data history generating process were constructed to 'prefer' a specific portfolio measure, subsequent analysis would be worthless. We address this problem by setting up a generating mechanism based on very few vital properties, which we assume to axiomatically hold true:

1. We assume the the existence of at least one shared risk factor driving (individual) line utilization. We identify this risk factor with the 'rating' of a customer, since various empirical studies show[37] that the rating is a valid driver of single credit line utilizations.[38] We acknowledge that financial institutions generally store more properties regarding individual credit lines, which could be employed to identify additional risk drivers.[39] The admittance of a single risk driver enables us to study the impact of different modeling techniques. To establish a flexible process for the

[36] The structure of this approach will be described in Section 4.2.

[37] See e.g. Agarwal *et al.* (2006), Sufi (2009) or Jiménez *et al.* (2009).

[38] Our approach is also valid for every other risk driver inherent in a portfolio of credit lines. Possible further risk factors that drive the utilization of credit lines can be (partially) covered by modeling an additional portfolio dependency structure. This approach will be detailed in Section 4.5.

[39] See e.g. Hubensack and Pfingsten (2010).

simulation of this risk driver, we will not rely on one specific model but rather implement *two different approaches* for the modeling of ratings in Section 4.3.

2. The proposed difference in individual draw-down behavior for each rating state is then established by assignment of *varying* marginal distributions. Conditional on the rating of the credit line at time t, individual draw-downs are simulated from two distributions (contingent on the credit line's rating), which are specified in Section 4.4.

3. We assume that not all interaction effects between different customers are covered through ratings. More precisely, we consider a portfolio dependency structure at the draw-down level. As it is again not clear how such a dependency structure best fits real data, we will describe a modeling approach for *different* structures, to account for unobservable effects in real data sets. The construction of a resulting portfolio draw-down distribution is finally discussed in Section 4.5.

Figure 1 illustrates the compilation of a draw-down history according to these three assumptions:

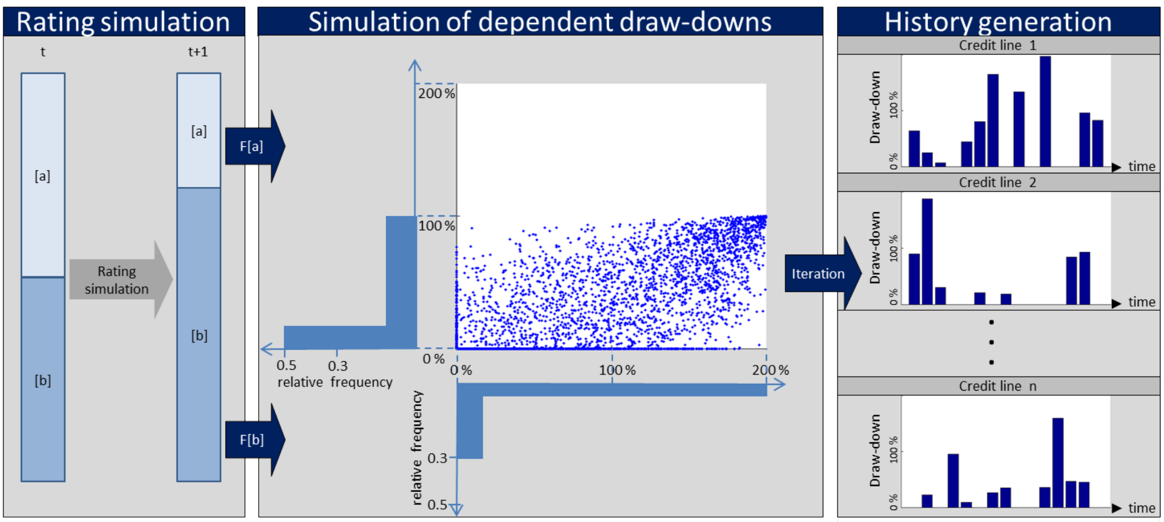

Figure 1: Illustration of the generating process for draw-down histories

We will introduce two setups for the simulation of customer ratings. The choices for marginal distributions as introduced in Section 4.4 are kept identical throughout all simulations. Further, we emphasize the importance of portfolio correlation. The approach to include such a structure, as outlined in Section 4.5, is therefore analyzed in some more depth, leaving us with four combinations. Overall we will simulate data histories for $2 * 4 = 8$ different environments for the generation of data histories. For the remainder of this paper, we refer to each of these eight combinations as an *environment* for the generation of portfolio draw-down histories.

4.2 General setup of the fictitious portfolio

We consider the following simplified scenario as a basic portfolio environment: A fixed number of $n = 250$ customers each have a credit line with granted volume of €1 at their disposal. Each customer i is associated with either a good credit quality ($r_{i,0} = [a]$) or a bad one ($r_{i,0} = [b]$). In each environment we assume 125 credit lines to start with rating $[a]$ and $[b]$, each. The time horizon is $T = 15$. This short observation window is directly reflected in the criterion for *dependency on provided data*: data histories from which vital model parameters need to be estimated are often rather short. A 'good' portfolio measure should therefore be tested on short data histories. The number of credit lines belonging to the portfolio is assumed to remain constant over time. No customers are allowed to leave the portfolio and no new customers are admitted.[40]

We assume the overall behavior of customers to not systematically vary over time. The structure of rating migrations, individual marginal distributions (contingent on a customer's rating) as well as the portfolio correlation structure introduced in the next 3 Subsections, do not change over time. Specifically, all stochastic properties in the portfolio are simulated exclusively based on the realizations of the portfolio in the former periods. Other than that, all properties are determined independently of the exact point in time.

4.3 Rating migrations

As lined out in Section 4.1, we will not rely on one model but rather implement *two different approaches* for the modeling of ratings.

[40] We acknowledge that in reality, customers would leave a credit lines portfolio and new customers would be added to a portfolio. However, at large banks it may be realistic that the overall portfolio structure and the risk attributes do not change to much over a short time horizon. Consequently, it may be a rough approximation that new customers would *only* replace customers leaving the portfolio.

We *first* model all ratings $r_{1,t}, ..., r_{n,t}$ for a fixed time t independently. The realization of each rating $r_{i,t}$ in this deterministic approach only depends on $r_{i,t-1}$ and a time-independent migration probability. Asides from the initial ratings $r_{1,0}, r_{2,0}, ..., r_{n,0}$, no other characteristics are available as input data in the model setup as described in Section 4.2. By providing a probability $p_{[a]\to[b]}(p_{[b]\to[a]}) \in [0,1]$ for a rating migration from rating 'class' $[a]$ ($[b]$) to $[b]$ ($[a]$), the rating migration behavior is completely determined. In Section 5, we will refer to this modeling approach as *deterministic*.

Secondly, for a more economically justified approach, we introduce a dependency structure between rating migrations of different customers. There exists a broad body of literature for the simulation of ratings on a portfolio basis, most prominent amongst which are simulations based on key risk factors driving a customers' creditworthiness.[41] The introduction of such a full factor model seems impractical, since we would need to provide further parameters such as factor weights. However, we apply a related model class for the simulation of rating migrations.

Considering that a customer can only migrate between two different rating classes, we need to determine the number of customer *leaving* each class at any time t. Such a two-state migration model can be implemented on the basis of a single factor risk model[42], where a migration into another (risk) state can simply be interpreted as a change in the realization of the underlying risk factor.

[41] See e.g. Bluhm *et al.* (2002) and Gupton *et al.* (1997).

[42] For an introduction of the single factor risk model framework, see Vacisek (1987). Since its implementation in *Credit Metrics*^{TM} through Gupton *et al.* (1997), the modeling of a single *Gaussian* risk factor has been established as an industry standard for the evaluation of credit portfolio risk. Conditions for the applicability of the large homogenous portfolio approximation (LHPA) outlaid in Vacisek (1987), are discussed in Gordy (2003). The one-factor Gaussian model has been criticized to misrepresent portfolio tail risk and may produce the implied correlation smile for CDO tranches when calibrated via market prices. For a unification of subsequent one factor models based on varying distributional assumptions, see Albrecher *et al.* (2007). A LHPA for this modeling class is discussed in the same work. However, as we are interested in simulating data histories for a range of plausible (basic) environmental assumptions, the rating migration approach presented in the present paper is based on the basic approximation from the *Credit Metrics*^{TM} framework.

For a 'sufficiently large' number of customers[43], the share of rating-changers can be approximated from the respective distribution function for an infinitely large portfolio:

$$F_{[a]\to[b]}(\omega) = \Phi\left[\frac{1}{\sqrt{\rho_{[a]}}}(\sqrt{1-\rho_{[a]}}\Phi^{-1}(\omega) - \Phi^{-1}(p_{[a]\to[b]})\right], \tag{14}$$

where Φ denotes the standard normal distribution function and $\rho_{[a]}$ is a correlation parameter. $F_{[b]\to[a]}$ is defined in analogy to (14).[44] Depending on the total number of customers with rating $[a]$ ($[b]$) at time t, the fraction of customers wandering off into rating class $[b]$ ($[a]$) at $t+1$ is simulated via equation (14). Fractions for both rating classes are determined independently and rounded to integers, such that the number of customers remains constant throughout the simulation. In Section 5, we refer to this modeling as *Vasicek approach.*

Migration probabilities $p_{[a]}$ and $p_{[b]}$ are set to 0.3 each, independent of the modeling approach. If the Vasicek approach is applied, the corresponding correlation parameters $\rho_{[a]}$ and $\rho_{[b]}$ are set to 0.05.

4.4 Marginal distributions

The marginal distributions $F_{[a]}$ and $F_{[b]}$ are chosen to reflect a difference in draw-down behavior associated with a customer's credit worthiness. Credit lines are often arranged as funding sources which are only utilized on an irregular basis (due to relatively high costs, as opposed to other funding sources). Assuming that customers with a good rating quality have access to alternative funding sources, these are assigned a relatively low average draw-down. We therefore assign high probabilities to a zero draw-down for each rating type. Furthermore, customer's with a 'bad' rating draw on their credit lines both more often and more severe, than customers with 'good' rating.[45]

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[43] The approximation approach via equation (14) may yield a less than optimal fit for the actual portfolio, due to a relatively small portfolio size of 250 credit lines. Nevertheless, we are not too concerned about the convergence of the migration distribution for a finite portfolio, against the limiting distribution stated in (14). Instead, the introduced approach is applied to simulate the modeling of rating migrations with an increased migration variability (as opposed to the deterministic approach introduced at the beginning of Section 4.3). A granularity adjustment applicable for the asymptotic single risk factor model is discussed in Lütkebohmert and Gordy (2007), but is not considered here.

[44] Additionally, we set $F_{[a]\to[a]} = 1 - F_{[a]\to[b]}$ and $F_{[b]\to[b]} = 1 - F_{[b]\to[a]}$.

[45] See Agarwal *et al.* (2006) and Norden and Weber (2010).

Probabilities for a zero draw-down amount to 0.5 and 0.3 for rating classes [a] and [b], respectively. The remaining probability mass is distributed equally to the interval $]0,1]$ for rating class [a] and the interval $]0,2]$ for rating class [b].[46]

4.5 Dependency structures

As we stated in Section 4.1, the model for generating draw-down histories is built on the assumption of different marginal draw-down distributions for a bank's customers, contingent on the rating. The simulation of different ratings therefore serves as a mechanism, by which a dependency structure between individual customers is transmitted to the draw-downs. Nevertheless, a setup which relies exclusively on the simulation of risk factors will never be able to capture all dependencies in a portfolio. Therefore, we account for possible 'leftover dependency' in the following way:

Based on the realizations from one of both rating approaches introduced in Section 4.3 (for a specific time t), we could simply draw a value for the actual, individual line draw-down from the corresponding marginal distribution to generate draw-downs $d_{1,t}, ..., d_{250,t}$. An independent draw-down then corresponds to a negligence of portfolio effects that are not captured by ratings alone. *Instead*, we first simulate realizations for a multivariate random variable $Y = (Y_1, ..., Y_n) \in [0,1]^n$, such that Y_i is equally distributed for every i. The dependency structure for Y is then transmitted to the individual credit line draw-downs through the application of the marginal distributions $F_{[a]}$ and $F_{[b]}$.

The simulation is implemented by application of copulas which produce the desired multivariate distribution. Specifically, we generate data histories based on correlation structures derived from (i) *Gaussian* as well as (ii) *t-copulas.* We apply a Gaussian copula due to its popularity in scholarly papers. The utilization of resulting multivariate normally distributed random variables for the *measurement* of various risks has been criticized in literature.[47] Furthermore, we use a t-copula as it can generate fatter tails than a Gaussian

[46] 'Equally' in this case refers to a vector containing $10,000$ entries for each distribution type. According to the rating type, $5,000$ or $3,000$ entries in the respective distribution vector are set to zero, whereas the remaining entries are filled according to the stated setup.

[47] For a discussion see Rachev *et al.* (2009).

copula.[48] Both copula types demand a positive (semi-)definite correlation matrix Γ as input parameter. An additional parameter η is required for the degrees of freedom applied within the t-copula simulation.

Based on Γ and (if required) η, we generate $T+1$ vectors $(y_{1,1}, ..., y_{n,1}) \in [0,1]^n, ...,$ $(y_{1,T+1}, ..., y_{n,T+1}) \in [0,1]^n$. Absolute draw-downs $d_{i,t}$ are derived by application of the respective marginal distribution: $d_{i,t} = \min(F^{-1}_{r_{i,t}}(y_{i,t}))$.

We apply *two different* correlation matrices Γ_1 and Γ_2 for the simulation of correlated draw-downs from the respective marginal distributions as the structure of a correlation matrices influences the portfolio draw-down distribution significantly. Considering that both matrices must contain $31,125$ non-trivial entries, we do not list Γ_1 and Γ_2 explicitly. Figure 2 shows histograms for the pairwise correlations contained in both matrices.

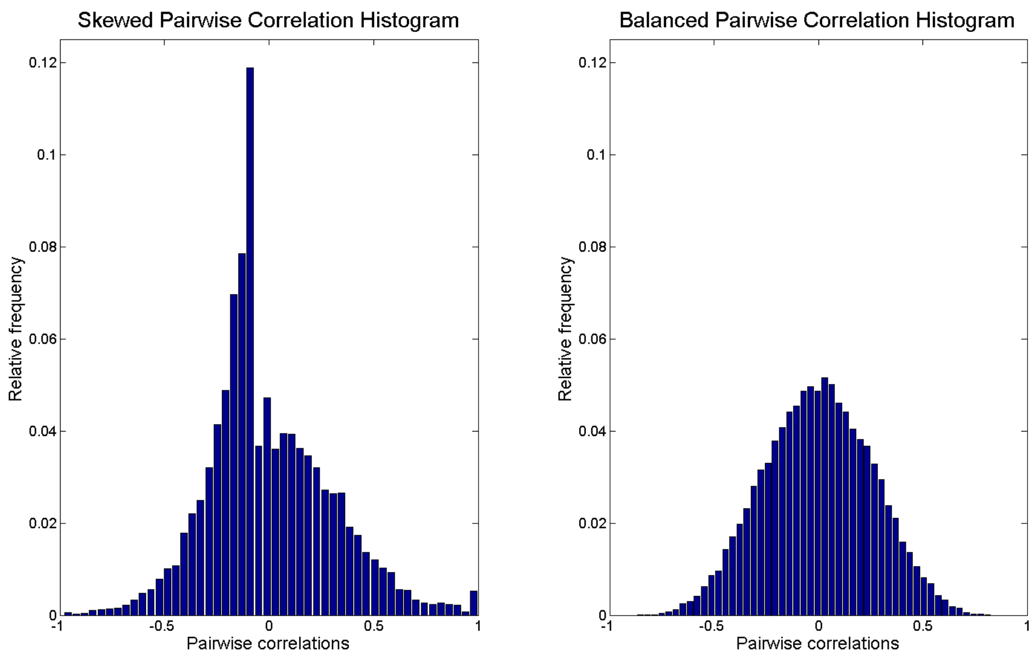

Figure 2: Pairwise correlation histogram

[48] See Didier *et al.* (2010) for a comparison of these copulas in a credit risk framework. Hamerle and Rösch (2005) analyze the effect of a misspecified copula in a latent variable credit risk framework. The authors apply a t-copula for the simulation of the correlation structure for the latent variable and base their estimate for the portfolio loss distribution on the deliberately wrongful assumption of an underlying Gaussian copula. Their analysis suggests that this misspecification of the copula type may result in large estimation errors for the true portfolio correlation, but continues to deliver adequate results for the estimated portfolio distribution.

Γ_1 is generated from a random sample of real credit lines.[49] In Section 5, we refer to this correlation structure as *skewed*. Γ_2 is generated randomly and will be referred to as *balanced* correlation type.

If the correlation structure is simulated on the basis of a t-copula, we set the parameter for degrees of freedom to $\eta = 7$.

[49] Sarin and Klein (2012) utilize a specific historical data set which we were also able to analyze. For a description of the data set, we refer to their work.

5 Monte Carlo Simulation results

In the following, we first specify the simulation setup in Section 5.1. Furthermore, we will also introduce benchmark portfolio measures using information regarding the respective environment as vital input for these measures. Consequently, we can analyze how 'good' a measures could perform, if all vital inputs are known rather than estimating them from 'historical' data.

In Section 5.2 we will analyze our funding measures on the basis of the criteria introduced in Section 3.

5.1 Description of the environment

The *eight* environments for the generation of artificial data histories differ according to the generating process for (i) the rating, (ii) the correlation structure between customers at the draw-down level and (iii) the applied copula type for the simulation of this correlation structure. Table 1 shows the coding for all eight environments, which are in the following applied for the simulation of draw-down histories:

assigned environment number	rating generation	correlation type	copula type
1	deterministic	skewed	Gaussian
2	deterministic	skewed	t
3	deterministic	balanced	Gaussian
4	deterministic	balanced	t
5	Vasicek	skewed	Gaussian
6	Vasicek	skewed	t
7	Vasicek	balanced	Gaussian
8	Vasicek	balanced	t

Table 1: Coding for all scenarios

Independent of the underlying environment, each single data history $(d_{i,t})_{1\leq i\leq n,1\leq t\leq T+1}$ contains draw-downs for $T+1=16$ points in time, from which only the first 15 dates are applied for the estimation of future draw-downs via the funding measures introduced in Section 2. The starting portfolio for all environments is set up in a symmetric fashion, i.e. at $t=0$, we start out with 125 customers in each rating class and equal migration probabilities $p_{[a]} = p_{[b]} = 0.3$. The characteristics of the environments determine the shape of the simulated data histories, which contain draw-downs for $t = 1, ..., 16$.

The evaluation of all introduced funding measures, is based on the simulated future draw-downs at $t = 16$. We simulate $M = 10,000$ data histories for all eight environments. Figure 3 shows a histogram of the resulting portfolio distribution at $t = T + 1 = 16$ for environment 1:

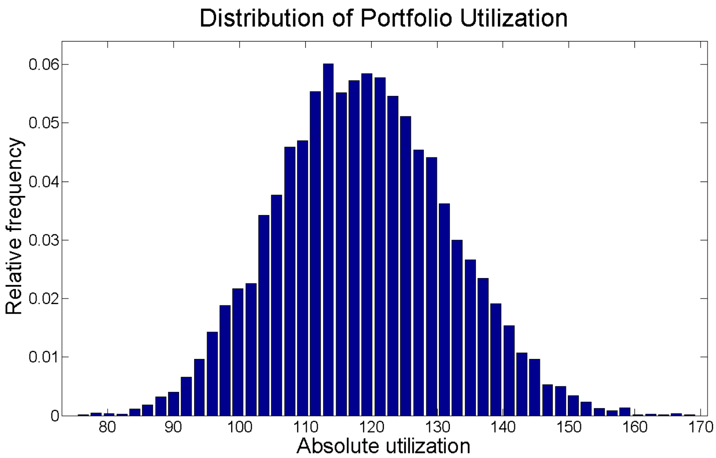

Figure 3: Monte-Carlo simulation results for the portfolio draw-down at $T + 1$

Given the data histories for each environment, we analyze the funding measures ν_{naive}, $\nu_{Heidorn}$, $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ based on the criteria derived in Section 3. For each data history H_m^S, another Monte-Carlo simulation with $10,000$ repetitions is conducted to determine values $\nu_{\hat{\sigma}}(H_m^S)$, $\nu_{95\%}(H_m^S)$ and associated ratios.

Finally, we want to identify exactly in how far the estimation of funding needs via $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ suffers from the necessity to estimate various parameters for the internal Monte-Carlo simulations. An *inclusion* of some of these parameters as model input for the estimation of a future draw-down distribution should result in a more exact funding measure. Of course, these funding measures have no practical relevance whatsoever, considering that such parameters are unknown in any realistic setting. However, we aim to identify which stochastic parameters are most problematic to estimate, in order to provide an indication for future enhancement potential for the respective estimation techniques.

The list of estimated properties for each environment S includes the marginal distributions $F_{[a]}$ and $F_{[b]}$, the correlation matrix Γ and the knowledge about the applied copula type (as well as the exact number of degrees of freedom $\eta = 7$ in the t-copula case). We calculate estimates for a measure $\nu_{95\%}^{comp}$, which follows the definition of $\nu_{95\%}$ with the exception, that all stochastic properties mentioned above are *included* as readily available input parameters for the internal Monte-Carlo simulation. We apply the same method to improve the performance for $\nu_{95\%}$, based on additional knowledge of the applied copula type and associated parameters (Γ and $\eta = 7$) alone. Results will be indicated via $\nu_{95\%}^{\Gamma}$. Finally, to control for the effect of a total negligence of correlation effects, we calculated future portfolio draw-downs assuming independent drawings from $F_{r_{i,T+1}}$ for all customers i, following the same logic.[50] Results will be indicated via $\nu_{95\%}^{indep}$.

5.2 Results

In the following, we compare the performances for all introduced portfolio measures ν_{naive}, $\nu_{Heidorn}$, $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ according to the criteria *exact determination of expected future draw-downs* (Section 5.2.1), *empirical shortfall probability* (Section 5.2.2), *processing of relevant information* (Section 5.2.3) and *dependency on provided data* (Section 5.2.4). As part of these analyses, we evaluate $\nu_{95\%}^{\Gamma}$, $\nu_{95\%}^{indep}$ and $\nu_{95\%}^{comp}$ to examine in how far the quality of the funding risk measure $\nu_{95\%}$ is driven by specific parameter estimations.

5.2.1 Exact determination of expected future draw-downs

To analyze the extent to which an exact determination of expected future draw-downs is accomplished, we calculate the expected deviation between estimated and realized portfolio draw-downs for all funding measures and for all eight environments. Naturally, only the estimated expected draw-downs are relevant for this criteria. Hence, differences in the estimation methods for contingent draw-downs do not matter for this analysis. Consequently, the estimations for ν_{naive} and $\nu_{Heidorn}$, as well as $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ must feature the same results, as $EDD_{\nu_{naive}} = EDD_{\nu_{Heidorn}}$ and $EDD_{\nu_{\hat{\sigma}}} = EDD_{\nu_{95\%}}$ by definition.

The varying setups for the rating generation, correlation structure and the copula type for each environment were established in a way, which does not systematically shift mean portfolio draw-downs over time. We therefore expect *all measures* to estimate expected future draw-downs (on average) correctly. Table 2 confirms this line of thought:

[50] This approach corresponds to the application of a unit matrix instead of $\hat{\Gamma}$ in Section 2.3.

	Environment S							
	1	2	3	4	5	6	7	8
ν_{naive}	0.02	−0.02	−0.05	−0.00	0.00	0.24	0.12	0.06
$\nu_{Heidorn}$	0.02	−0.02	−0.05	−0.00	0.00	0.24	0.12	0.06
$\nu_{\hat{\sigma}}$	0.00	−0.05	−0.06	−0.03	−0.06	0.19	0.07	0.02
$\nu_{95\%}$	0.22	−0.00	−0.06	−0.03	−0.06	0.19	0.07	0.02
$\nu_{95\%}^{indep}$	0.00	−0.05	−0.06	−0.03	−0.05	0.19	0.07	0.02
$\nu_{95\%}^{\Gamma}$	0.00	−0.05	−0.06	−0.03	−0.06	0.19	0.07	0.02
$\nu_{95\%}^{comp}$	0.03	0.03	−0.10	0.03	0.01	0.23	0.12	0.05

Table 2: Simulation results for $\mathbb{E}_S(EDD_\nu - \widetilde{PF_S})$

Considering that the mean (absolute) portfolio draw-down lies in the range of 120 for all scenarios,[51] these results can be interpreted as stochastic fluctuations and should converge to zero while increasing the number of employed data histories M beyond $10,000$. Hence, the expected future draw-downs are estimated quite exact for *all measures.*

Furthermore, we evaluate the absolute deviations from expected future draw-downs for all funding measures in all environments. Table 3 shows the results:

	Environment S							
	1	2	3	4	5	6	7	8
ν_{naive}	10.71	13.73	9.31	12.83	12.99	15.49	11.94	14.70
$\nu_{Heidorn}$	10.71	13.73	9.31	12.83	12.99	15.49	11.94	14.70
$\nu_{\hat{\sigma}}$	10.65	13.67	9.23	12.77	12.62	15.12	11.54	14.30
$\nu_{95\%}$	10.65	13.67	9.23	12.77	12.62	15.12	11.54	14.30
$\nu_{95\%}^{indep}$	10.65	13.66	9.23	12.77	12.62	15.12	11.54	14.30
$\nu_{95\%}^{\Gamma}$	10.65	13.67	9.23	12.77	12.62	15.11	11.54	14.30
$\nu_{95\%}^{comp}$	10.00	12.97	8.58	12.02	9.94	12.99	8.52	12.01

Table 3: Simulation results for $\mathbb{E}_S(|EDD_\nu - \widetilde{PF_S}\,|)$

Results indicate only small differences for the performances of all measures. Nevertheless, the measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ seem to consistently estimate expected future draw-downs more exactly, compared to ν_{naive} and $\nu_{Heidorn}$. The setup of the benchmark measure $\nu_{95\%}^{comp}$ includes information for all model parameters of the data generating process. Comparing the aforementioned differences to the 'lower boundary', which is provided by

[51] For a rough approximation of this property, one can simply calculate the mean expected draw-down for a time horizon of one time step, with respect to the marginal distributions $F_{[a]}$ and $F_{[b]}$: $125 * (0.5 * 0 + 0.5 * \frac{1-0}{2}) + 125 * (0.3 * 0 + 0.7 * \frac{2-0}{2}) = 118.75$.

$\mathbb{E}_S(|EDD_{\nu_{95\%}^{comp}} - \widetilde{PF_S}\ |)$, indicates that on average expected future draw-downs are indeed captured more precisely through $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$, i.e. the average absolute deviations are smaller for $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ than for ν_{naive} and $\nu_{Heidorn}$.

Moreover, the differences $\mathbb{E}_S(|EDD_{\nu_{95\%}} - \widetilde{PF_S}\ |) - \mathbb{E}_S(|EDD_{\nu_{95\%}^{comp}} - \widetilde{PF_S}\ |)$ are highest, if the Vasicek rating approach is applied (environment $5-8$), as illustrated in Table 4:[52]

	Environment S							
	1	2	3	4	5	6	7	8
Differences	0.65	0.70	0.65	0.75	2.68	2.13	3.02	2.29

Table 4: Results for $\mathbb{E}_S(|EDD_{\nu_{\hat{\sigma}}} - \widetilde{PF_S}\ |) - \mathbb{E}_S(|EDD_{\nu_{95\%}^{comp}} - \widetilde{PF_S}\ |)$

Due to the fact that we applied a very simplistic rating simulation approach within the calculation of $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$, which does not capture the dependency structure inherent in the Vasicek rating generation, this decrease in estimation performance is expected. On the other hand, the differences $\mathbb{E}_S(|EDD_{\nu_{naive}} - \widetilde{PF_S}\ |) - \mathbb{E}_S(|EDD_{\nu_{95\%}} - \widetilde{PF_S}\ |)$ and $\mathbb{E}_S(EDD_{\nu_{Heidorn}} - \widetilde{PF_S}\ |) - \mathbb{E}_S(|EDD_{\nu_{95\%}} - \widetilde{PF_S}\ |)$ are even higher for the Vasicek scenarios, indicating that considering the most recent portfolio ratings *at all* can deliver a better estimate.

Contrarily, modeling the portfolio dependency structure via a t-copula (scenarios 2, 4, 6 and 8) does not increase these differences, although the optimal prediction quality via $\nu_{95\%}^{comp}$ decreases considerably.

Furthermore, the measure $\nu_{95\%}^{indep}$, which *completely ignores* the underlying portfolio correlation structure, shows (almost) the same results as $\nu_{95\%}$. This finding again confirms intuition, according to which expected future draw-downs should not depend on portfolio correlation effects.[53]

5.2.2 Empirical shortfall probability

The probability according to which a measure's predicted funding needs will not suffice to satisfy customers' liquidity demands at $t = T + 1$ is of utmost importance for the liquidity management of the entire credit line portfolio. Only if the shortfall probability

[52] The illustrated differences apply to both measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$, since $EDD_{\nu_{\hat{\sigma}}} = EDD_{\nu_{95\%}}$.

[53] A direct application of calculation rules for expectation values shows this as well.

associated with a measure's predicted expected and contingent future draw-downs is in line with the risk tolerance of the portfolio manager, this measure will be utilized. A basic condition for such a comparison naturally lies in a measure's ability to capture a relatively constant shortfall probability via its funding needs, *over all environments.* Table 3 shows the simulation results for the empirical shortfall probabilities $\mathbb{P}_S(\nu - \widetilde{PF}_S < 0)$ for all environments S and all funding measures:

	Environment S							
	1	2	3	4	5	6	7	8
ν_{naive}	18.05%	17.76%	17.61%	17.59%	17.78%	17.56%	18.03%	18.10%
$\nu_{Heidorn}$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
$\nu_{PF,\sigma}$	13.97%	19.24%	13.21%	20.01%	17.2%	20.55%	17.82%	22.29%
$\nu_{PF,95\%}$	3.14%	7.7%	3.01%	8.44%	6.18%	9.45%	6.51%	10.90%
$\nu_{PF,95\%}^{indep}$	14.05%	19.67%	10.55%	18.06%	17.24%	21.24%	15.49%	20.49%
$\nu_{PF,95\%}^{\Gamma}$	5.75%	5.27%	5.58%	5.33%	8.83%	6.91%	9.73%	7.62%
$\nu_{PF,95\%}^{comp}$	5.18%	4.58%	5.13%	5.07%	5.19%	4.74%	4.79%	4.96%

Table 5: Simulation results for $\mathbb{P}_S(\nu - \widetilde{PF}_S < 0)$

By definition high shortfall probabilities result from low (absolute) values for ν. This relationship is confirmed in the results for mean total deviations between estimated and realized total draw-downs:

	Environment S							
	1	2	3	4	5	6	7	8
ν_{naive}	12.65	16.31	10.98	15.20	15.17	18.66	14.05	17.49
$\nu_{Heidorn}$	166.74	166.51	166.82	166.56	166.61	166.71	166.83	166.51
$\nu_{\hat{\sigma}}$	14.79	15.00	13.08	13.43	14.83	15.37	13.34	13.59
$\nu_{95\%}$	25.27	25.64	22.24	22.80	25.37	26.09	22.59	23.03
$\nu_{95\%}^{indep}$	14.78	14.71	14.71	14.73	14.70	14.95	14.83	14.78
$\nu_{95\%}^{\Gamma}$	21.59	29.14	18.69	26.94	21.52	29.35	18.80	26.94
$\nu_{95\%}^{comp}$	20.94	28.93	17.97	26.67	20.90	29.13	18.06	26.68

Table 6: Simulation results for $\mathbb{E}_S(\nu - \widetilde{PF}_S)$

As expected, the risk measure introduced by Heidorn *et al.* (2008) largely overestimates the realized portfolio draw-downs. Whereas a zero shortfall probability *for every environment* may technically not sound too bad, the fact that $\nu_{Heidorn}$ overestimates the actual draw-down *on average* by more than €166.00 (which amounts to more than half of the nominal portfolio value), certainly does.

On the other hand, the 'simple' estimation approach ν_{naive} misses the target value of a 5% shortfall by more than 12 percentage points in all simulations. Considering that this measure has not been calibrated according to this target in any way, the relatively constant shortfalls realized through ν_{naive}'s application lead to the assumption that a massive increase in performance regarding criterion 2 may be achieved through a recalibration.

The same recalibration methodology could (potentially) be applied to derive better results for $\nu_{\hat{\sigma}}$. Both measures (ν_{naive} and $\nu_{\hat{\sigma}}$) address contingent future draw-downs by means of a parameter α. A very simple recalibration approach can be established upon the assumption of normally distributed portfolio draw-downs, with some parameters μ_{PF} and σ_{PF}. In this case, 95% of the total probability mass for this distribution would be covered by calibrating contingent draw-downs through $\alpha = \Phi^{-1}(0.95)$, assuming that the portfolio standard deviation σ_{PF} for both measures is estimated somewhat correctly. Calculation of the respective shortfall probabilities for this approach, based on the data histories that were simulated for the analysis, led to very constant shortfalls in the range of $7\% - 8\%$.[54] However, it is a priori not plausible to approximate portfolio draw-downs by means of a normal distribution.

Shortfall values for $\nu_{95\%}$ fluctuate around the target value of 5%. Since this portfolio measure allows the addressing of a specific default probability by definition, we examine the associated results for all environments in some more detail by comparison with the results for $\nu^{\Gamma}_{95\%}$, $\nu^{indep}_{95\%}$ and $\nu^{comp}_{95\%}$:

- Estimating future funding needs via $\nu_{95\%}$ based on a data history generated through a t-copula approach obviously poses some problems, as indicated by the overestimations of the shortfall probabilities for $\nu_{95\%}$ in the environments 2, 4, 6 and 8. Additionally, the good performance of $\nu^{comp}_{95\%}$ in comparison to the benchmark $\nu^{\Gamma}_{95\%}$ for all environments indicates that a more precise estimate may be obtained, if the the underlying dependency structure (i.e., the copula type) is known. We acknowledge, that the Gaussian copula approach applied in the calculation of $\nu_{95\%}$ is a far from optimal 'fit' for an underlying t-copula. A more precise estimation of the exact

54 An additional calculation of $\mathbb{P}_S(\nu_{naive} - \widetilde{PF}_S < 0)$ and $\mathbb{P}_S(\nu_{\nu_{\hat{\sigma}}} - \widetilde{PF}_S < 0)$ for $\alpha = 1.64 \approx \Phi^{-1}(0.95)$ yields these results.

dependency structure via clustering specific segments of the data set may therefore improve performance regarding this issue. Nevertheless, estimating the dependency structure through the introduced (Gaussian) copula approach clearly yields superior results, as compared to a total negligence of a non-trivial dependency structure, which is obvious from the very bad results for $\nu_{95\%}^{indep}$.

- A comparison of the results for $\nu_{95\%}^{\Gamma}$ and $\nu_{95\%}^{comp}$ in the environments 1, 3, 5 and 7 reveals overestimations of the shortfall probabilities for $\nu_{95\%}^{\Gamma}$. The only estimated properties in the model for $\nu_{95\%}^{\Gamma}$ are the rating generation process and the marginal distributions $F_{[a]}$ and $F_{[b]}$. As the estimation of both marginal distributions is very straight-forward, we expect main shortcomings of $\nu_{95\%}^{\Gamma}$ to result from the estimation of future ratings. We therefore compare the associated results in the environments 1 and 3 (deterministic rating) to the results for the environments 5 and 7 (Vasicek model). The results for the environments 5 and 7 indicate that the Vasicek model is indeed captured less adequately as the overestimation of shortfall probabilities for $\nu_{95\%}^{\Gamma}$ in these environments is more severe, as compared to the shortfall probabilities in the environments 1 and 3 (by more than 1 percentage point). The applied Vasicek model introduces correlations between customers belonging to the same rating class at any time t. The funding measures $\nu_{\hat{\sigma}}$, $\nu_{95\%}$, $\nu_{95\%}^{\Gamma}$, $\nu_{95\%}^{indep}$ and $\nu_{95\%}^{comp}$ estimate ratings via a 'deterministic' approach, ignoring possible correlations and therefore misestimating resulting tail risks. Robustness simulations have shown that decreasing the Vasicek correlation parameters $\rho_{[a]}$ and $\rho_{[b]}$ weakens this effect. A sound portfolio model that is calibrated on an actual data set should reflect rating migrations in a more realistic way by application of a factor model.[55]

- Furthermore, we study the impact of estimating the correlation structure via $\hat{\Gamma}$. Therefore, we compare the results for $\nu_{95\%}$ and $\nu_{95\%}^{\Gamma}$ in the environments 1, 3, 5 and 7.[56] We apply $\nu_{95\%}^{\Gamma}$ as a benchmark, since this funding measure utilizes the correlation structure as given input, whereas the remaining properties are estimated as usual. In all four environments, the shortfall probability is underestimated by $\nu_{95\%}$ as compared to $\nu_{95\%}^{\Gamma}$. As outlined in Section 4.5, a correlation structure for the funding measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ is estimated for each draw-down history. Intuitively one suspects short data histories for (stochastically) independent customers to pro-

[55] This method is broadly used in Credit Risk Models (see Bluhm *et al.* (2002)).

[56] We focus on the Gaussian environments, as bad estimates for the other environments supposedly result from the poor performance in estimating the underlying t-copula through a Gaussian approach, rather than a misestimation of the correlation matrix Γ.

duce rather large empirical correlation coefficients.[57] This effect may lead to the estimation of a correlation matrix $\hat{\Gamma}$ which shows a higher volatility for its entries, than the original matrix Γ. The application of such a matrix may in turn lead to a higher resulting correlation in the portfolio, in turn demanding a higher liquidity reserve and consequently underestimating the shortfall probability. Furthermore, the estimations for a 'skewed' correlation structure in the data are of the same quality as for a 'balanced' correlation structure.[58]

5.2.3 Processing of relevant information

The inclusion of additional portfolio information in the form of customer ratings (for the calculation of $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$) is only justified, if it can be employed to derive a better estimate. Specifically, if such information is applied 'correctly', the estimate should fluctuate less volatile around the true portfolio draw-down at $t = T + 1 = 16$. Table 7 shows the results for the standard deviation of the difference between estimated and realized total portfolio draw-downs, for all portfolio measures in all data environments:

	Environment S							
	1	2	3	4	5	6	7	8
ν_{naive}	13.66	17.69	11.85	16.51	16.59	19.85	15.26	18.88
$\nu_{Heidorn}$	15.44	19.77	13.27	18.42	17.97	21.67	16.18	20.35
$\nu_{\hat{\sigma}}$	13.36	17.22	11.56	16.07	15.88	19.04	14.51	18.05
$\nu_{95\%}$	13.45	17.35	11.62	16.15	15.98	19.15	14.57	18.14
$\nu_{95\%}^{indep}$	13.38	17.21	11.60	16.10	15.91	19.04	14.58	18.08
$\nu_{95\%}^{\Gamma}$	13.33	17.12	11.55	15.97	15.87	18.97	14.51	17.98
$\nu_{95\%}^{comp}$	12.49	16.21	10.73	15.04	12.45	16.23	10.71	15.02

Table 7: Simulation results for $\sqrt{\mathbb{V}_S(\nu - \widetilde{PF})}$

In all environments, the estimation of future funding needs through $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ exhibits a lower volatility, compared to the first two approaches.

$\nu_{Heidorn}$ clearly fluctuates more strongly than all other measures. As we laid out in Section 2.2, this measure is exclusively calibrated via a quantile derived from the historical distribution of customer draw-downs, and therefore ultimately neglects most properties of

[57] *Large* concerns positive as well as negative deviations.

[58] This conclusion follows from comparing results for environments 1 and 5 ('skewed' correlation structure) with results for environments 3 and 7 ('balanced' correlation structure).

that distribution. Also, customer-specific information (i.e., the rating) is not regarded at all. The bad results for $\nu_{Heidorn}$ again indicate a clear miscalculation of future portfolio funding needs, *even as compared to* ν_{naive}, which simply considers the portfolio's historical volatility.

Both measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ apparently have weaknesses regarding the estimate's preciseness, if a t-copula is applied for the generation of the underlying data histories (environments $2, 4, 6$ and 8).[59] On the other hand, results for $\nu_{95\%}^{comp}$ indicate that the application of a t-copula ultimately leads to a less precise estimate, compared to the utilization of a Gaussian copula for the data history generation. Table 8 lists the differences of the portfolio measures $\nu_{95\%}^{\Gamma}$ and $\nu_{95\%}$ to the benchmark measure, based on the results from table 7:

	Environment S							
	1	2	3	4	5	6	7	8
$\nu_{\hat{\sigma}}$	0.87	1.01	0.83	1.03	3.43	2.81	3.80	3.03
$\nu_{95\%}$	0.96	1.14	0.89	1.11	3.53	2.92	3.86	3.12

Table 8: Results for $\mathbb{E}_S(|\nu - \nu_{95\%}^{comp}|)$

The calculated differences do not systematically increase, if the t-copula approach is regarded. Instead, these differences are clearly more pronounced in the last 4 scenarios, where ratings were simulated on the basis of the Vasicek approach. As we mentioned in section 5.2.2, this problem may be addressed, in an actual portfolio environment, by application of a more precise rating factor model.

5.2.4 Dependency on provided data

The calculation of ν_{naive} as well as $\nu_{Heidorn}$ strongly depends on an extensive history of portfolio observations as contingent future draw-downs are estimated, based on historical portfolio fluctuations. If the data history for the assessment of these fluctuations is very short, the quality of the estimate should decline rapidly. Moreover, anecdotal evidence suggests that data sets containing information on credit line utilization are usually incomplete,[60] resulting in a need to estimate missing values, from which further errors may

[59] Considering that the estimation via the control measure $\nu_{95\%}^{\Gamma}$ exhibits the same weakness, problems arising from the estimation of the 'true' correlation matrix Γ can be ruled out as the main reason for these shortcomings.

[60] See Heidorn *et al.* (2008), p.17.

follow.

The measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ are clearly less susceptible to incomplete data as well as to short data histories: For $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ it is necessary to derive estimates for $p_{[a]\to[b]}, p_{[b]\to[a]}$, $F_{[a]}$, $F_{[b]}$ and Γ. An appropriate estimation may also be derived from data concentrating on a short time horizon or, at least for the marginal distributions, even for a single point in time. Of course, the estimate for the 'true' correlation matrix Γ is (much) more precise, if a long data history is available. In Section 5.2.2 we discussed that the utilization of a (moderately) incorrect correlation matrix poses less of a problem than fitting an underlying t-copula via a Gaussian approach.

Poor data *quantity* can be compensated by applying external estimates for the model parameters for customers with comparable characteristics (this applies for $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$).[61] Contrarily, the measures ν_{naive} and $\nu_{Heidorn}$ strictly depend on the quality of the underlying data set and cannot be improved by external data.

5.3 Summary of the results

The results from Section 5.2.1 show that all four measures ν_{naive}, $\nu_{Heidorn}$, $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ capture expected future draw-downs very exactly for all environmental settings. However, comparing the measures in terms of the empirical shortfall probability (Section 5.2.2), processing of relevant information (Section 5.2.3) and dependency on provided data (Section 5.2.4), we conclude that $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$ outperform the 'historical' funding measures ν_{naive} and $\nu_{Heidorn}$. Furthermore, an extension of this modeling approach to employ realistic data more efficiently by including additional risk drivers or estimating future ratings through a (multi-)factor model, is straight-forward. The development a more profound estimation method for the assessment of overall portfolio correlation, leaves further room for improvement. Comparing measures $\nu_{\hat{\sigma}}$ and $\nu_{95\%}$, we further propose the utilization of the latter measure, as it offers a clearer estimate for total downside funding risk.

Summarizing the results, a forecast model of type $\nu_{95\%}$ in our study dominates the other three funding measures to aforementioned quantitative as well as economic benefits.

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61 I.e., by utilization of external rating matrices.

6 Conclusion

The comparison of portfolio measures' performance for the management of funding risk inherent in portfolios of credit facilities is non-trivial. To address this topic, we set up an environment for the simulation of fictional data histories on the basis of few natural assumptions *and* derived desirable properties, a 'good' portfolio funding risk measure should meet.

The data generating mechanism is constructed around a number of key characteristics, including the simulation of customer ratings and the application of an additional portfolio dependency structure. Data histories are simulated for eight different environments, allowing us to study various portfolio measures for varying environmental assumptions.

We discuss several quantitative measures *and* propose the utilization of a Monte-Carlo simulation approach, which yields viable results even for supposedly extreme data environments.

Our work opens further areas for research. The dependency structure between draw-downs of single credit lines is difficult to estimate, in particular for short observation histories. In addition, various building blocks in the estimation of future portfolio funding needs via $\nu_{95\%}$ can be enhanced. These include especially the application of a (multi-)factor model for the rating process and the inclusion of additional risk factors. Research in this area might further improve the performance of the introduced Monte-Carlo simulation approach.

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