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On the Compatibility of Value at Risk, Other Risk Concepts, and Expected Utility Maximization

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On the Compatibility of Value at Risk, Other Risk Concepts, and Expected Utility Maximization

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Abstract

Value at risk is en vogue as an internal measure of risk which is also accepted by regulators. The paper provides a critical appraisal based on comparisons with the related concepts of stochastic dominance and lower partial moments. In particular, we find that value at risk contradicts second-degree stochastic dominance and thus expected utility maximization for non-satiated, risk averse individuals. This might lead to regulation having adverse effects on banks' risk taking.

1 Introduction

The Capital Adequacy Directive (CAD) issued by the European Union (EU) and the „Basle Capital Accord to Cover Market Risk“ (Basle Accord) issued by the Bank for International Settlements (BIS) limit banks' risk taking via a bound on the risk/equity ratio. Risk may either be calculated by in-house models or by certain standard methods.

For making investment decisions, however, banks may have another notion of risk. If the notion of risk contained in the regulation and that used by banks in their investment decisions differ, it is not clear what the effects of the regulatory limit on a bank's risk position will be for the bank's risk taking. Particularly, it seems possible that with differing notions of risk the regulatory limit might in some cases cause banks to take a riskier position than they would without any limit.

In order to derive statements about the effects of regulatory capital requirements on banks' risk taking, we need to first clarify the relation between different concepts of risk, which is the main purpose of this paper. The effects on risk taking will then be studied in a companion paper.

We start from a normative rule for decisions under uncertainty, which is „maximize expected utility“. This is presented in Section 2. The literature knows concepts of risk which are compatible with expected utility maximization, namely second-degree stochastic dominance and mean preserving spreads. These concepts are presented in Section 3. In Section 4, we ask if the variance and the value at risk are also compatible with expected utility maximization. The results from Section 3 are used as a benchmark. In Section 5, we look at a class of risk measures - lower partial moments - which are also candidates for risk measures compatible with expected utility maximization. Section 6, then, concludes our paper.

2 Decisions under Uncertainty

To obtain a benchmark for the comparison of different concepts of risk, we ask what notion of risk banks use in their investment decisions. We will answer this question normatively rather than empirically; i.e. we say what notion of risk banks *should* use, rather than actually *do* use.

Banks' investment decisions are decisions under uncertainty. If we accept the axioms put forward by von Neumann and Morgenstern¹, then we know that a rational investor should make his investment decisions in such a matter that he maximizes his expected utility.

For our discussion, we suppose that investment opportunities (securities) are described by the set of possible payoffs in different potential future states of nature and the probabilities of these payoffs; i.e. every security is described by a density function of future payoffs.² We are arguing within a one-period model. Also, we assume the securities' density functions to be completely specified and known.

1 Cf. von Neumann/Morgenstern (1953).

2 We assume continuous distributions. The following thoughts can be transferred to discrete distributions by replacing densities with probability functions and adjusting the concepts accordingly.

Normatively, the bank chooses the securities such as to maximize expected utility. Since one bank's utility function³ will most likely be different from another bank's, it is not always possible to identify one portfolio that is optimal for every bank or every investor for that matter.⁴ However, the concept of efficient portfolios enables us to identify the subset of all conceivable portfolios, which contains the optimal portfolios of all investors.

Commonly made, though not universally accepted, assumptions about investors' utility functions are:

- 1) The utility functions are increasing, i.e. investors prefer more wealth to less.
- 2) The utility functions are concave, i.e. investors are risk averse.

An alternative idea is to choose securities according to two characteristics of their payoff distributions - risk and expected return. In this paper, we are interested only in the risk component. Facing two securities or portfolios of securities that have the same expected return, a risk averse investor will choose the security or portfolio that is less risky.

We thus have two statements about an investor's optimal choice between two portfolios with the same expected return (i.e. between two density functions with the same mean):

- a) Choose the portfolio that is less risky.
- b) Choose the portfolio that has the higher expected utility.

We know that b) constitutes rational behavior. The question we are interested in is: What definition of „less risky“ ensures that a) also constitutes rational behavior. That is, what concepts of risk⁵ are compatible with maximizing expected utility, and especially, is value at risk?

From the literature it is known that the concepts of stochastic dominance of different degrees, which we will introduce next, are compatible with maximizing expected utility for certain classes of utility functions.

3 The fact that we are talking about *the* bank's utility function is based on the assumption that there are no principal-agent conflicts and that the bank exhibits monocentric decision making.

4 Note, however, that under conditions implying Tobin separation, the composition of risky portfolios will be identical across investors. The capital asset pricing model is one important case in point (cf. Brealey/Myers (1996), p. 179-184).

5 We will not examine „*downside* risk“ in the sense of Menezes et al. (1980) which gives importance to third-order derivatives of utility functions.

3 Second-Degree Stochastic Dominance

3.1 Definition

The concept of stochastic dominance⁶ is a means for partially ordering density functions. Stochastic dominance utilizes the complete information contained in the density function, whereas other concepts, such as for example the mean-variance approach, condense the information of the density function into a few parameters (μ and σ^2 for the mean-variance approach), thus potentially forfeiting information.

The basic idea in tying together the comparison of two portfolios according to the concept of stochastic dominance with maximizing expected utility is that investors prefer low payoffs to have low probabilities and high payoffs to have high probabilities.⁷ This assumes that investors prefer more wealth to less wealth. In terms of utility functions: Investors' utility functions $U(x)$, where x is the future payoff, are increasing: $\frac{dU}{dx} > 0$. Using different forms of the concept of stochastic dominance for maximizing expected utility for *all* investors one makes different additional assumptions about investors' utility functions.

As we have stated above, we make the additional assumption that investors are risk averse. This means that all investors have concave utility functions: $\frac{d^2U}{dx^2} < 0$. If investors have increasing concave utility functions, then second-degree stochastic dominance (SSD) is a necessary and sufficient condition for higher expected utility for *all* investors.⁸

Let $f(x)$ and $g(x)$ be two density functions. $F(x)$ and $G(x)$ represent their respective cumulative distribution functions (c.d.f.s). The density functions are defined on the closed intervall $[c,d]$, where $c < d$.⁹ For SSD, f is said to stochastically dominate g if from c up to every value $y \in [c,d]$ the area under G is greater than or equal to the area under F (cf. Figure 1):

6 An extensive bibliography on stochastic dominance, listing about 400 publications, is found in Bawa (1982).

7 Cf. Joy/Porter (1974), p. 26.

8 For a proof see Hadar/Russell (1969, 1971), Hanoch/Levy (1969), Rothschild/Stiglitz (1970).

9 Cf. Hadar/Russell (1971), p. 289.

$$\int_c^y G(x) dx \geq \int_c^y F(x) dx, \forall y \in [c, d].^{10}$$

This is equivalent to:

$$(1) \quad \int_c^y [G(x) - F(x)] dx \geq 0, \forall y \in [c, d].$$

f strictly dominates g if the inequality strictly holds for at least one value of y.¹¹

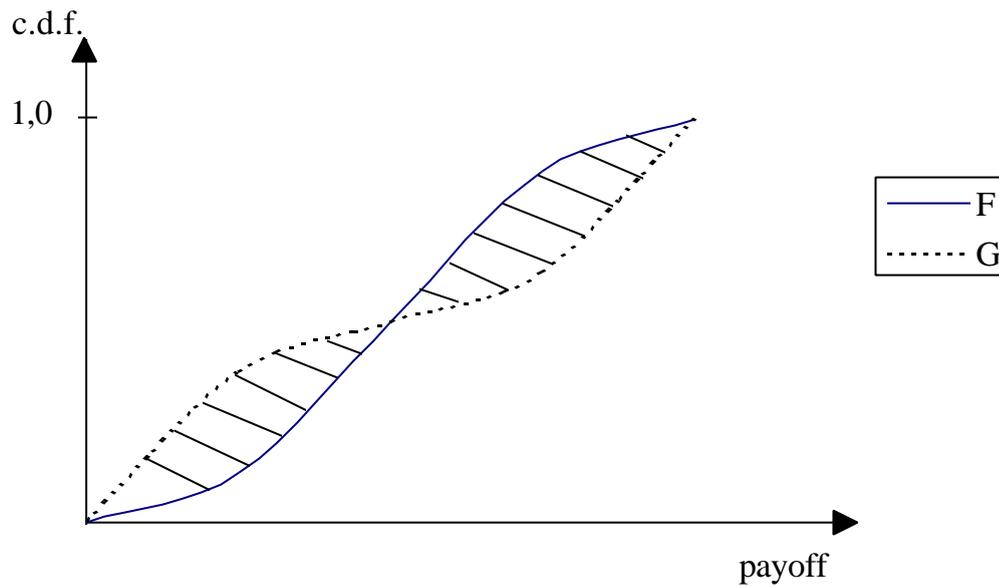


Figure 1: f shows SSD over g

SSD is only a partial ordering. For example, the two c.d.f.s shown in Figure 2 cannot be ordered according to second-degree stochastic dominance.

¹⁰ Cf. Hadar/Russell (1969), p. 27.

¹¹ Cf. Hadar/Russell (1971), p. 289.

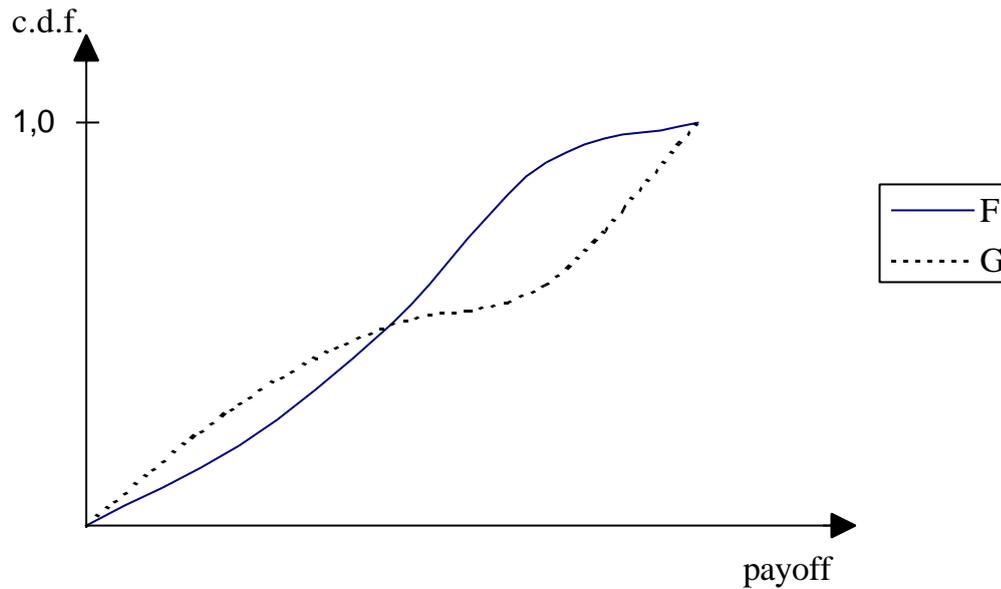


Figure 2: f and g cannot be ordered by SSD

A necessary condition for SSD of f over g is that the expected value (the mean) of f is greater than or equal to the expected value of g:

$$\int_c^d f(x) x dx \geq \int_c^d g(x) x dx. \quad 12$$

Since we are interested in differences in risk only, we select f and g to have the same mean, i.e.:

$$\int_c^d f(x) x dx = \int_c^d g(x) x dx .$$

3.2 Second-Degree Stochastic Dominance and Mean Preserving Spreads

If f and g have the same mean, then SSD of f over g is equivalent to the Rothschild/Stiglitz notion of „g having more weight in the tails than f“¹³.

g having more weight in the tails than f means that g can be obtained from f by a sequence of mean preserving spreads:

12 Cf. Hanoch/Levy (1969), p. 340.

13 Cf. Rothschild/Stiglitz (1970), p. 226.

$$g(x) = f(x) + s(x),$$

where $s(x)$ is the sequence of mean preserving spreads. The corresponding c.d.f. is:

$$(2) \quad S(x) = G(x) - F(x).$$

In a mean preserving spread, probability weight is taken away from two intervals of the density function and added to two different intervals. The probability weight taken from the left one of the two intervals is shifted further to the left, and the probability weight taken from the right one of the two intervals is shifted further to the right.

Rothschild/Stiglitz show that g can be obtained from f by a sequence of mean preserving spreads if and only if two conditions (called the integral conditions) hold:

$$C1: \int_c^d S(x) dx = 0^{14}$$

$$C2: \int_c^y S(x) dx \geq 0, \forall y \in [c, d].$$

Condition C1 ensures that f and g have the same mean. Using (2) condition C2 can also be written as

$$\int_c^y [G(x) - F(x)] dx \geq 0, \forall y \in [c, d].$$

This expression is identical to expression (1), i.e. the condition that characterizes SSD of f over g .

3.3 Second-Degree Stochastic Dominance and White Noise

In the same paper, Rothschild/Stiglitz show¹⁵ that the partial ordering of densities according to the integral conditions is equivalent to the partial ordering achieved by the statement that g is riskier than f , if g is equal to f plus noise.

Consider two random variables X_A and X_B , the distribution of X_A being $f(x)$ and the distribution of X_B being $g(x)$ such that

$$X_B = X_A + Z.$$

14 For convenience, Rothschild/Stiglitz set $c=0$ and $d=1$.

15 Cf. Rothschild/Stiglitz (1970), p. 237 - 240.

If Z is a random variable with an expected value of zero¹⁶ which is uncorrelated with X_A , then Z is called white noise and g is riskier than f .

3.4 The Equivalences

Thus, we know that the partial preference orderings that can be obtained by the concept of second-degree stochastic dominance and the two presented Rothschild/Stiglitz statements are all compatible with the decision principle of maximizing expected utility as long as investors exhibit increasing concave utility functions.

In our context of the search for rational measures of risk, we know that the following definitions of risk (that are all equivalent) are compatible with maximizing expected utility for increasing concave utility functions:

g is riskier than f if

- g and f have the same mean and f shows second-degree stochastic dominance over g ,
- g has more weight in the tails than f ,
- g is equal to f plus noise.

However, since these are partial orderings only, in a lot of cases these concepts of risk will not enable one to decide whether g is riskier, less risky or equally risky compared to f . Also, these concepts of risk only rank risk, they do not give an absolute value for the amount of risk encountered. Divisibilities of investment opportunities and the resulting multitude of portfolios which would have to be compared under the SSD criterion, render the SSD concept impractical for asset allocation decisions. That is why specific indices such as for example the variance and the value at risk are used. Our next step, therefore, is to look at the compatibility of the measures variance and value at risk with the three concepts of risk described above. Since these three concepts which are compatible with maximizing expected utility are all equivalent, we will use just one of them, second-degree stochastic dominance, as our benchmark.

4. Specific Risk Measures and their Compatibility with Expected Utility Maximization

¹⁶ From $E(Z)=0$ it follows that $E(X_B)=E(X_A)$, i.e. X_A and X_B have the same mean.

4.1 Compatibility of Variance and Second-Degree Stochastic Dominance¹⁷

Since the variance is a generally known measure, we will not give a definition. Let f and g have the same mean. If g is riskier than f according to SSD, then g also has a greater variance than f . The opposite does not hold, however: If g has a greater variance than f , then g is not necessarily riskier than f in the sense of SSD. The reason being that SSD is only a partial ordering. That is, if g has a greater variance than f , then either

- a) g is riskier than f in the sense of SSD or
- b) g and f cannot be compared under SSD.

Let „ $g \overset{r}{>}_{s^2} f$ “ denote „ g is riskier than f according to the variance“ and „ $g \overset{r}{>}_{SSD} f$ “ „ g is riskier than f according to second-degree stochastic dominance“, then:

$$g \overset{r}{>}_{SSD} f \Rightarrow g \overset{r}{>}_{s^2} f, \text{ but}$$

$$g \overset{r}{>}_{s^2} f \not\Rightarrow g \overset{r}{>}_{SSD} f.$$

4.2 Compatibility of Value at Risk and Second-Degree Stochastic Dominance

Value at risk is the notion of risk embodied in the Basle Accord. The value at risk is the loss that with a certain one-sided confidence level will not be exceeded over a certain time period. In our context, the value at risk of a security or portfolio is the loss that will be exceeded only with a certain probability (usually 5% or 1%) according to the given density function.

Measuring the magnitude of a loss for a given payoff distribution requires knowledge of the initial investment amount (I). A density's value at risk with a certain confidence level [$VaR(p)$] equals the initial investment amount minus the density function's p -quantile [$Q(p)$], where p is 1 minus the chosen level of confidence:

$$VaR(p) = I - Q(p), \text{ where } \Pr(X \leq Q(p)) = p.$$

$\Pr(\cdot)$ denotes the probability of the event in parantheses.

¹⁷ For proofs of this relationship between the variance and SSD (or equivalently the Rothschild/Stiglitz notion of risk) see for example: Rothschild/Stiglitz (1970), p. 241-242; Huang/Litzenberger (1993), p. 49.

For simplicity, we assume the initial investment amount to be zero. The value at risk then equals the negative of the density function's p-quantile. The p-quantile in turn is the inverse function of the cumulative distribution function F:

$$\text{VaR}(p) = -Q(p) = -F^{-1}(p).$$

According to the value-at-risk concept of risk, the higher the value at risk of an investment opportunity (a density function of payoffs) is, the riskier it is.

Like the variance, value at risk delivers a complete ordering. We have seen above that the ordering of riskiness achieved by the variance is compatible with that achieved by second-degree stochastic dominance, except for those cases where the densities considered cannot be ordered by SSD. Next, we will ask if for densities with equal means a corresponding relationship exists between value at risk and SSD.

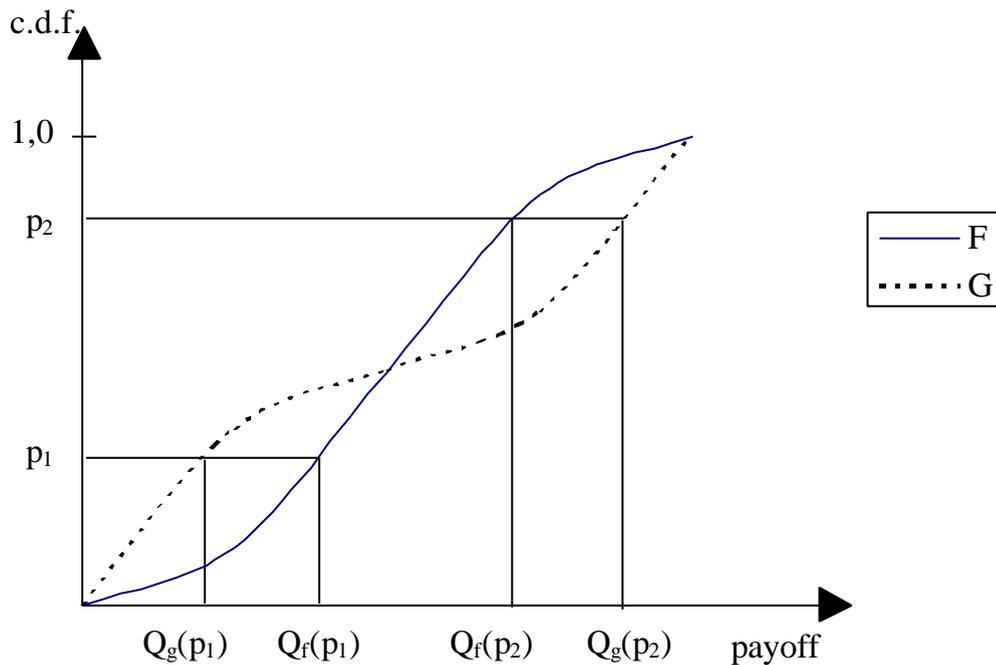


Figure 3: SSD and VaR

Figure 3 depicts two densities, f and g , which have the same mean. The density function f dominates g according to second-degree stochastic dominance, i.e. g is riskier than f according to SSD: $g \overset{r}{\underset{SSD}{>}} f$.

As we have seen above, the value at risk is the negative of the p -quantile, where p is *one* specific value. First, let us pick p to be p_1 . From Figure 3, we see that $Q_g(p_1) < Q_f(p_1)$. Since $VaR(p) = -Q(p)$, this is equivalent to $VaR_g(p_1) > VaR_f(p_1)$. Thus, according to value at risk (with $p = p_1$) g is riskier than f : $g \overset{r}{>}_{VaR(p_1)} f$. This is the same conclusion that is reached using SSD.

Next, let us pick p to be p_2 . From Figure 3, we see that now $Q_g(p_2) > Q_f(p_2)$, which is equivalent to $VaR_g(p_2) < VaR_f(p_2)$. Thus, if we set $p = p_2$, then according to value at risk f is riskier than g : $f \overset{r}{>}_{VaR(p_2)} g$. This is the opposite of the conclusion that is reached using SSD.

Note that contradicting risk ranking also results at the point where the two cumulative distribution functions intersect. Since at this point the value at risks for f and g are identical, f and g would be considered of the same risk according to the value-at-risk concept.

We draw three conclusions:

- 1) The risk ranking of investment opportunities (densities) according to the value-at-risk concept is not independent of the chosen confidence level.
- 2) The risk ranking of investment opportunities (densities) according to the value-at-risk concept may contradict the risk relation given by the SSD concept (our benchmark):

$$g \overset{r}{>}_{SSD} f \not\Rightarrow g \overset{r}{>}_{VaR(p)} f, \text{ and correspondingly}$$

$$g \overset{r}{>}_{VaR(p)} f \not\Rightarrow g \overset{r}{>}_{SSD} f.$$
- 3) Investment decisions (in the sense of determining the efficient set) based on the expected return and the value at risk would not necessarily maximize expected utility and are thus not rational.

4.3 The Special Case of Normally Distributed Portfolio Payoffs

As we have seen with $p=p_1$, if at point p the c.d.f. of g lies above the c.d.f. of f , then and only then: $g \stackrel{r}{SSD} > f \Rightarrow g \stackrel{r}{VaR(p)} > f$. It is frequently assumed - though it is not empirically sustainable - that the densities of portfolios' payoffs are normally distributed.¹⁸ The peculiarity of the comparison of the c.d.f.s of two normal distributions with equal means is that these c.d.f.s intersect with each other only once. And this point of intersection is exactly known, since each normal distribution has a cumulative probability of 0.5 when it reaches its mean because of the symmetry of the distribution.

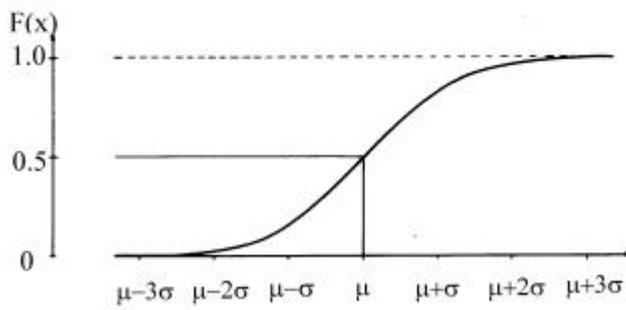


Figure 4: Cumulative distribution function for the normal distribution

Figure 4 shows the general shape of a normal distribution's cumulative distribution function $F(x)$. We see that for all normal distributions $F(\mu) = 0.5$. In addition, we see that $F(x)$ takes on fixed values for μ minus any positive multiple of the standard deviation σ and also takes on fixed values for μ plus any positive multiple of σ .

A normal distribution is completely described by its two moments expected value μ and standard deviation σ . For our purposes, the expected values are always identical. If the standard deviations were identical, too, we would have identical densities. Conversely, whenever the densities are not identical, their standard deviations must differ.

¹⁸ We have assumed at the outset of this paper that the density functions are defined on the closed interval $[c,d]$, where $c < d$. For normally distributed densities, we need to assume that the densities are defined on the open interval $]-\infty, \infty[$.

Let f and g be two non-identical normally distributed densities with equal means ($\mu_f = \mu_g = \mu$). f and g then have different standard deviations. μ minus any positive multiple of the standard deviation of f is always different from μ minus that positive multiple of the standard deviation of g , and μ plus any positive multiple of the standard deviation of f is also always different from μ plus that positive multiple of the standard deviation of g . Thus, except for $F(\mu) = G(\mu) = 0.5$, $F(x)$ and $G(x)$ do not have any common point.

If the standard deviation of g is greater than that of f , then up to μ , G lies above F , because μ minus a multiple of the standard deviation of g is always less than μ minus that multiple of the standard deviation of f , and to the right of μ , G lies below F , because μ plus a multiple of the standard deviation of g is always more than μ plus that multiple of the standard deviation of f .

We thus know that the c.d.f.s of two normal distributions with equal means cross only once, and do so at $x = \mu$, where the c.d.f.s take on the value 0.5. For any payoff lower than μ , we know that the c.d.f. of the normal distribution with the higher standard deviation (G) lies above the c.d.f. of the normal distribution with the lower standard deviation (F). For any payoff higher than μ , G lies under F . From the symmetry of the normal distribution it also follows that the area between G and F from minus infinity to μ and the area between G and F from μ to plus infinity are of the same size. Thus, we know that f shows second-degree stochastic dominance over g . Note that from the arguments above it follows that two normal distributions with equal means can always be ordered by SSD, i.e. for normal distributions with equal means SSD is a complete ordering.

Therefore, if two densities are normally distributed with equal means and the confidence level $(1-p)$ used in computing value at risk is fixed by the regulatory authority at more than 50 % (i.e. $p < 0.5$), then $g \overset{r}{SSD} > f \Rightarrow g \overset{r}{VaR(p)} > f$ ($\forall p < 0.5$); thus, for normal distributions the ordering of riskiness delivered by the value-at-risk concept is compatible with that delivered by SSD.

For normal distributions the converse also holds: $g \overset{r}{VaR(p)} > f \Rightarrow g \overset{r}{SSD} > f$ ($\forall p < 0.5$), because two normal distributions with equal means can always be ordered by SSD.

Whenever we cannot be sure, however, that at the confidence level used the c.d.f. of the SSD-dominated alternative lies above the c.d.f. of the SSD-dominant one, the risk ranking given by value at risk may contradict that given by SSD. This will typically be the case, since actual distributions of portfolios' payoffs may be asymmetric (for example due to options) and are not necessarily of the same type.

4.4 Compatibility of Value at Risk and First-Degree Stochastic Dominance

We have seen in the preceding section that contradictions in the risk assessment between value at risk and SSD may arise if c.d.f.s cross. Therefore, it seems plausible that risk assessment according to value at risk does not contradict a stochastic-dominance concept which only orders c.d.f.s that do *not* cross: first-degree stochastic dominance (FSD).

In order for the concept of first-degree stochastic dominance to be compatible with expected utility maximization, the only required assumption about investors' utility functions is that they are increasing (cf. p. 4). In contrast to the second-degree stochastic dominance concept, this not only encompasses risk averse investors, but also those who are risk neutral or risk seeking.

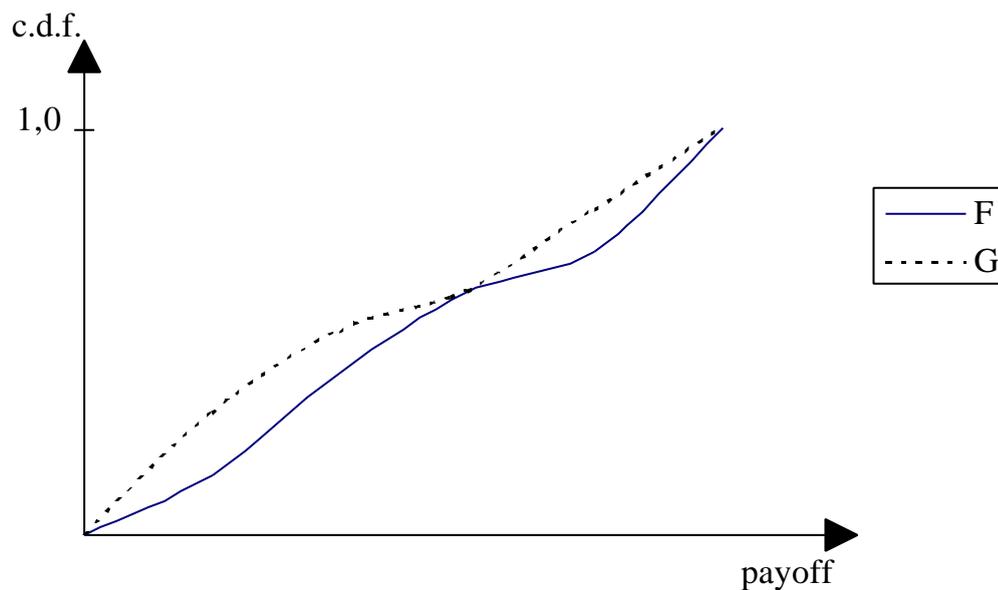


Figure 5: f shows FSD over g

According to FSD, f is said to stochastically dominate g if G always lies on or above F :

$$G(x) \geq F(x), \forall x \in [c,d].$$

f strictly dominates g if the inequality strictly holds for at least one value of x .¹⁹

If investors have increasing utility functions, first-degree stochastic dominance is a necessary and sufficient condition for higher expected utility for *all* investors.²⁰

Figure 5 shows the c.d.f.s of two densities, f and g , that can be ordered according to first-degree stochastic dominance. Each investor who prefers more wealth to less chooses portfolio f over g , because F is never above and at least for one payoff below G .

Let us assume that the regulatory authority stipulates a one-sided confidence level of one minus p_1 . From Figure 6, we see that $Q_g(p_1) < Q_f(p_1)$, which is equivalent to $VaR_g(p_1) > VaR_f(p_1)$. Thus, according to value at risk (with $p = p_1$) g is riskier than f :
 $g \underset{VaR(p_1)}{>} f$.

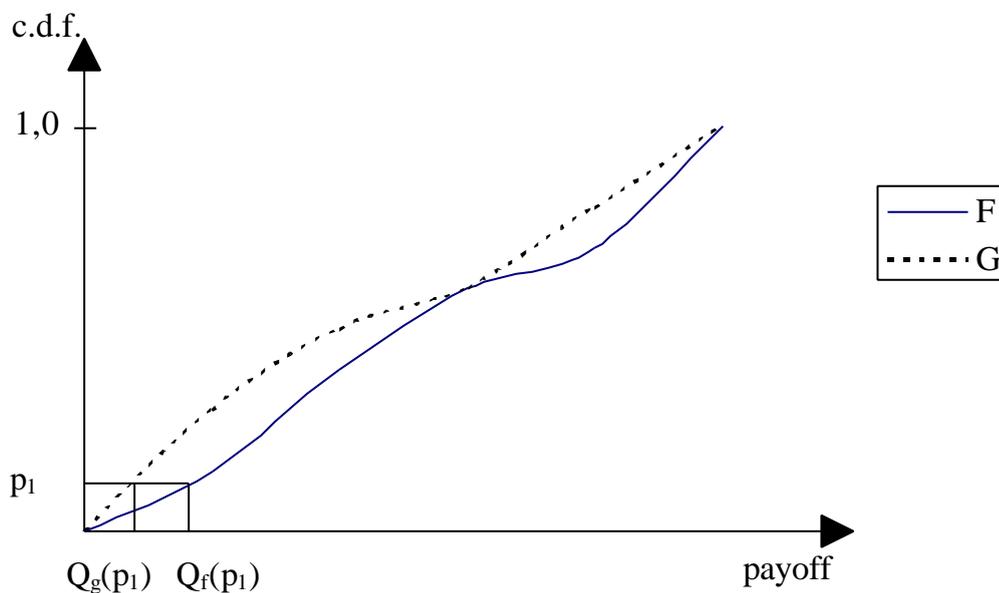


Figure 6: FSD and Value at Risk

¹⁹ Cf. Hadar/Russell (1971), p. 289.

²⁰ For a proof see Quirk/Saposnik (1962), Hadar/Russell (1969, 1971), Hanoch/Levy (1969).

In the beginning of our paper, we argued that an investor's choice of portfolio depends on the portfolios' expected returns and risks. In order to find a measure of risk that is compatible with maximizing expected utility, we isolated the risk component by setting expected returns equal. Unfortunately, two c.d.f.s with identical means can never be ordered by FSD, because they either cross or are identical. Thus, we cannot derive a risk ranking from the FSD-concept. In Figure 6, f shows first-degree stochastic dominance over g . Since f has a higher expected return than g however, we cannot claim that from f being stochastically dominant it follows that f is less risky.

Thus, we cannot make a statement about the compatibility of risk assessment according to the value-at-risk concept and the FSD-concept. What we can say is that if f shows first-degree stochastic dominance over g , then the value at risk of f is smaller than that of g , *except* for those p where the two c.d.f.s touch. At those points where the two c.d.f.s touch, the value at risks of f and g are identical. Contrast this observation with Figure 3, where the densities f and g can be ordered by SSD, but not by FSD. There, p_1 and p_2 yield opposite rankings according to value at risk.

5 Lower Partial Moments

5.1 Definition

The value at risk is only based on some lower part of the density considered. This is also true for lower partial moments. To compute a lower partial moment, the investor selects some value as the target payoff, denoted by t , for his portfolio investment, and then for any portfolio, he looks at those payoffs (with their corresponding probabilities) that fail to meet his target, i.e. he looks at that part of the density function of the portfolio's payoff that lies to the left of his target.

The definition of the lower partial moment of order n with target t ($LPM_n(t)$) is

$$LPM_n(t) = \int_c^t (t - x)^n f(x) dx. \quad ^{21}$$

The higher n , the higher is the weight put on the largest negative deviations from the target in computing the lower partial moment.

$LPM_1(t)$, for example, gives the expected amount by which the target is missed (the expected shortfall). For all payoffs above the target, the target is reached and the shortfall in these cases is therefore zero.

21 Cf. Fishburn (1977), p.116.

5.2 Lower Partial Moment Zero and Value at Risk

For $n = 0$ we get:

$$\begin{aligned} \text{LPM}_0(t) &= \int_c^t (t - x)^0 f(x) dx \\ &= \int_c^t 1 \cdot f(x) dx = F(t) - F(c) = F(t) - 0 \\ &= F(t). \end{aligned}$$

Now, suppose we choose a certain confidence level $(1 - p)$ and compute the corresponding value at risk. Next, we set the investor's target to zero (the initial investment amount) minus this value at risk. We then get

$$\text{LPM}_0(-\text{VaR}(p)) = F(-\text{VaR}(p))$$

which is equivalent to

$$\text{VaR}(p) = -F^{-1}(\text{LPM}_0(-\text{VaR}(p))),$$

provided the inverse of F exists.

This shows that the value at risk can be transformed into the lower partial moment zero. The lower partial moment zero for the target $t = -\text{VaR}(p)$ gives the probability that the actual loss will be greater than $\text{VaR}(p)$. From the definition of value at risk (cf. p. 9-10) we know that $\text{VaR}(p) = -F^{-1}(p)$. $\text{LPM}_0(-\text{VaR}(p))$ thus is equal to p , i.e. one minus the level of confidence with which the value at risk is computed.

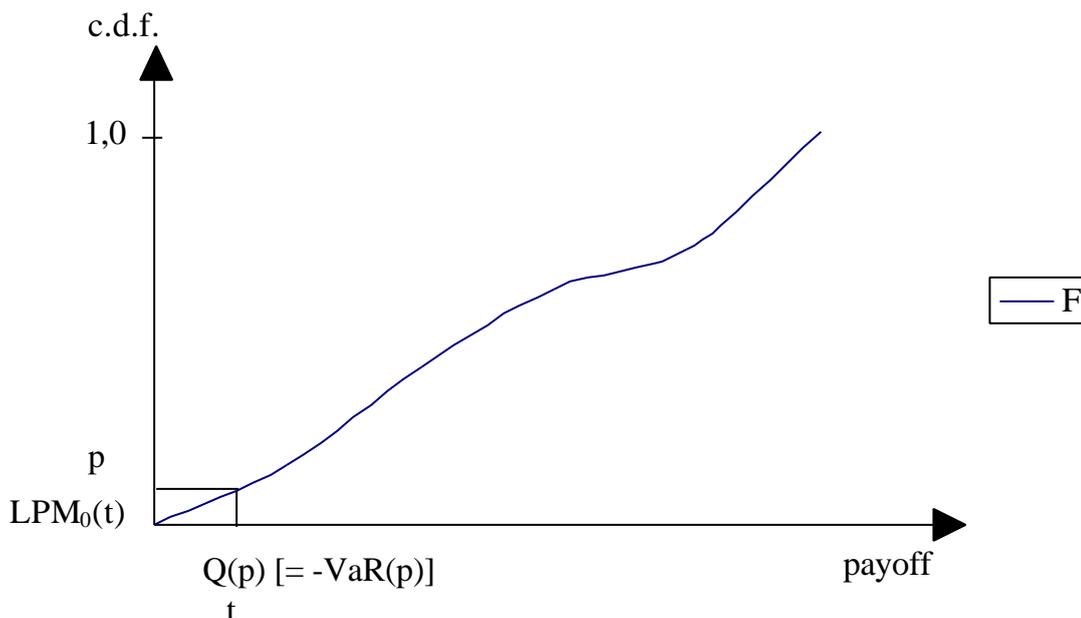


Figure 7: VaR and LPM_0

Figure 7 further illustrates that $LPM_0(-VaR(p))$ and $VaR(p)$ look at the same point on the cumulative distribution function - only from different axis (remember that $VaR(p) = -Q(p)$).

5.3 Lower Partial Moments and Stochastic Dominance in General²²

Fishburn (1977) has examined the general relationship between lower partial moments and stochastic dominance. He collects the following results:

- If f shows first-degree stochastic dominance over g (f FSD g), then the lower partial moment of g of any order ($LPM_n(t)_g$) is higher than the corresponding lower partial moment of f , except for the lower partial moment zero at those points (targets) where the two c.d.f.s touch²³:

$$f \text{ FSD } g \Rightarrow LPM_n(t)_g > LPM_n(t)_f \quad \forall t, \text{ except for } [n = 0 \wedge F(t) = G(t)].$$

- Particularly, if f shows first-degree stochastic dominance over g , then the lower partial moment zero of g is higher than the lower partial moment zero of f , except for those targets where the two c.d.f.s touch:

$$f \text{ FSD } g \Rightarrow LPM_0(t)_g > LPM_0(t)_f \quad \forall t, \text{ except where } F(t) = G(t).$$

Because of the relation of the lower partial moment zero and the value at risk discussed above, this means that if f shows first-degree stochastic dominance over g , then the value at risk of g is higher than the value at risk of f , except for those confidence levels where the two c.d.f.s touch (cf. p. 16):

$$f \text{ FSD } g \Rightarrow VaR_g(p) > VaR_f(p) \quad \forall p \in [0,1], \text{ except where } F^{-1}(p) = G^{-1}(p).$$

In other words, as long as the value at risks are not equal, a lower value at risk is a necessary condition for first-degree stochastic dominance.

²² Note that in this paragraph we are considering the general relationship between lower partial moments of densities and the densities' stochastic dominance relations. That is, in this paragraph we are not assuming that the densities have identical means.

²³ This exception is not made in Fishburn's statement (cf. Fishburn (1977), p. 123). However, as we have argued on page 16, it is necessary, because at this point the value at risks and thus the lower partial moments zero of f and g are equal.

- If f shows second-degree stochastic dominance over g (f SSD g), then the lower partial moment of g of any order higher than zero (and thus *not* the value at risk, which refers to the lower partial moment zero) is higher than the corresponding lower partial moment of f , except when f and g have the same mean (μ) and the c.d.f.s of n -th order of f and g are equal at the target considered:

$$f \text{ SSD } g \Rightarrow LPM_n(t)_g > LPM_n(t)_f \quad \forall n \geq 1, \quad \forall t,$$

except for $[\mu_f = \mu_g \wedge F_n(t) = G_n(t)]$,

where F_n and G_n are the n -th order cumulative distribution functions.

- For $i \geq 1$ the following relationship holds between i -th degree stochastic dominance (i SD) and lower partial moments:²⁴

$$f \text{ } i\text{SD } g \Rightarrow LPM_n(t)_g > LPM_n(t)_f \quad \forall n \geq i-1, \quad \forall t,$$

except for $[n \geq 1 \wedge \mu_f = \mu_g \wedge F_n(t) = G_n(t)]$

and except for $[n = 0 \wedge F(t) = G(t)]$.

5.4 Lower Partial Moment One and Second-Degree Stochastic Dominance

Of the relations between lower partial moments and stochastic dominance described above, one is of particular interest to us:

$$f \text{ SSD } g \Rightarrow LPM_1(t)_g > LPM_1(t)_f \quad \forall t,$$

except for $[\mu_f = \mu_g \wedge F(t) = G(t)]$.

That is, if f shows second-degree stochastic dominance over g , the lower partial moment one of g is higher than the lower partial moment one of f ; except when f and g have the same mean and the c.d.f.s of f and g are equal at the target considered.

As we have stated before, in order to isolate the densities' risks, we select f and g to have the same mean. Then, except for those targets at which the c.d.f.s of f and g are equal, the following statement holds:

If g is riskier than f according to second-degree stochastic dominance, then g also has a greater lower partial moment one than f . The converse again is not true, because SSD is only a partial ordering.

24 Cf. Fishburn (1977) and Bawa (1978).

$$g \underset{SSD}{\overset{r}{>}} f \Rightarrow g \underset{LPM_1(t)}{\overset{r}{>}} f, \quad \forall t, \text{ except for } F(t) = G(t), \text{ but}$$

$$g \underset{LPM_1(t)}{\overset{r}{>}} f \not\Rightarrow g \underset{SSD}{\overset{r}{>}} f.$$

Thus, except for those cases where the densities cannot be ordered by SSD and also except for those targets where the c.d.f.s are equal, the ordering of riskiness achieved by the lower partial moment one is compatible with that achieved by SSD and thus with maximizing expected utility.

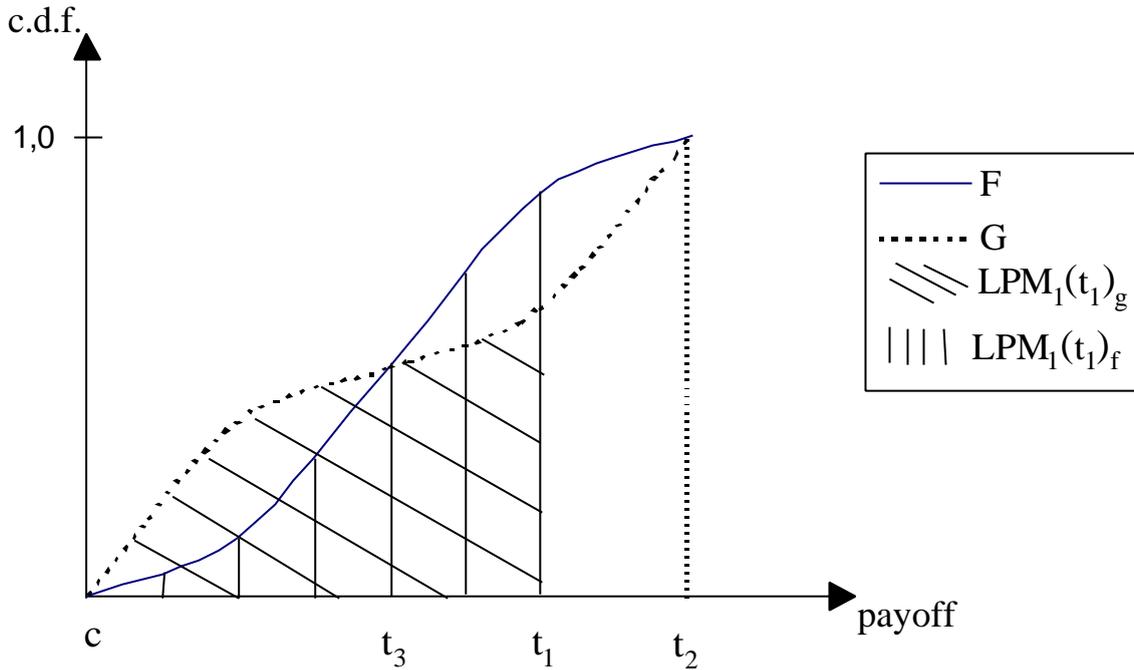


Figure 8: SSD and Lower Partial Moment One (LPM_1)

Figure 8 illustrates this. The density function f shows second-degree stochastic dominance over g , because from the very left (the starting point of both c.d.f.s) up to every payoff the area under G is greater than or equal to the area under F , and at least at one payoff (e.g. at t_1) the area under G is strictly greater than the area under F .

Thus: $g \underset{SSD}{\overset{r}{>}} f$.

The lower partial moment one of a given target is equal to the area under the c.d.f. up to that target. Consider the target payoff t_1 . From Figure 8, we can see that the area under G up to t_1 is greater than the area under F up to t_1 . Thus:

$LPM_1(t_1)_g > LPM_1(t_1)_f$. According to lower partial moment one, g is riskier than f : $g \underset{LPM_1(t_1)}{>^r} f$, which is the same risk ranking as that given by SSD.

For SSD of f over g , the area under G from the very left (c) up to *every* payoff must be greater than or equal to the area under F from c up to that payoff. For $LPM_1(t)_g > LPM_1(t)_f$, the area under G from c up to the *one single* target t considered must be greater than the area under F from c up to that target. SSD of f over g is thus sufficient, but not necessary for $LPM_1(t)_g > LPM_1(t)_f$, except for those payoffs/targets where the areas are equal. This further clarifies that except for those payoffs where the areas are equal, f SSD g implies $LPM_1(t)_g > LPM_1(t)_f$.

If f shows SSD over g and \bar{t} is some payoff, then the area below F all the way up to \bar{t} can be at most equal to that below G . Suppose they are indeed equal. Then, for continuous c.d.f.s F and G , we must have $F(\bar{t}) = G(\bar{t})$:

- For $F(\bar{t}) > G(\bar{t})$, there would exist some small $\varepsilon > 0$ such that $F(\bar{t} + \delta) > G(\bar{t} + \delta)$ for all δ such that $0 \leq \delta \leq \varepsilon$. Therefore, the area below F up to $(\bar{t} + \varepsilon)$ would be greater than that below G , contradicting SSD of f over g .
- For $F(\bar{t}) < G(\bar{t})$, there would exist some small $\varepsilon > 0$ such that $F(\bar{t} - \delta) < G(\bar{t} - \delta)$ for all δ such that $0 \leq \delta \leq \varepsilon$. Therefore, the area below F up to $(\bar{t} - \varepsilon)$ would be greater than that below G , again a contradiction.

In Figure 8, $F(\bar{t}) = G(\bar{t})$ holds for $t = c$, $t = t_2$ and $t = t_3$. For $t = c$ and $t = t_2$, the areas under F and G up to \bar{t} are equal. Thus, $LPM_1(c)_g = LPM_1(c)_f$ and $LPM_1(t_2)_g = LPM_1(t_2)_f$ meaning $g \underset{LPM_1(c)}{=}^r f$ and $g \underset{LPM_1(t_2)}{=}^r f$, although we know from the SSD-concept that g is indeed riskier than f .

At target t_3 , too, the two c.d.f.s are equal, i.e. $F(t_3) = G(t_3)$. However, the area under G from the very left up to t_3 is greater than the corresponding area under F , i.e. $LPM_1(t_3)_g > LPM_1(t_3)_f$. This correctly identifies g as the riskier density: $g \underset{LPM_1(t_3)}{>^r} f$.

Thus, for $F(t) = G(t)$, compatibility between the SSD and the $LPM_1(t)$ risk assessment is not generally given, but for some t (t_3 in Figure 8) it *may* still hold.

As long as $F(t) \neq G(t)$, however, as we have seen, if g is riskier than f according to the SSD-concept, g 's higher risk will correctly be identified by the lower partial moment one.

6 Conclusion

We have argued that, for non satiated and risk averse investors, ranking the riskiness of two portfolios with the same expected return according to the concept of second-degree stochastic dominance is compatible with maximizing expected utility. The variance as a risk measure is in turn compatible with the SSD risk assessment except for those cases where the portfolios cannot be ordered according to SSD.

Value at risk as a risk measure on the other hand was found to potentially contradict the SSD risk assessment. Compatibility of value at risk and stochastic dominance (again only for those portfolios that can be ordered by the stochastic dominance concept) can only be found for first-degree stochastic dominance, which does not utilize the assumption of risk aversion.

It was then argued that value at risk corresponds to the lower partial moment zero. With a related measure, the lower partial moment one, a risk measure that is compatible with second-degree stochastic dominance was found. Again, the compatibility is restricted to those portfolios which can be ordered by SSD. In addition, the compatibility holds only for those targets used in calculating lower partial moments one for which the portfolios' cumulative distribution functions are not equal.

The in-house models which banks may use for calculating their risk positions according to the Basle Accord build on the concept of value at risk, which has been shown to be incompatible with second-degree stochastic dominance. Regulation may thus eliminate portfolios from a bank's choice set which are SSD-dominant, and at the same time not eliminate those portfolios which are SSD-dominated by those eliminated portfolios. This may in some cases lead the bank to choose a riskier portfolio under regulation than it would choose in the absence of regulation; where „riskier“ means riskier in the SSD-sense. This is equivalent to saying that a bank might choose one certain portfolio in the absence of regulation, while under regulation it might choose another portfolio which differs from the first by a series of mean preserving spreads, i.e. which has more probability weight in the tails. We elaborate on these potential adverse effects which the Basle Accord may have on banks' risk taking in a companion paper.

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