Put Options Are Not Too Expensive

— An Analysis of Path Peso Problems —

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Abstract

The observed prices of out-of-the money put options seem too high given standard derivative pricing models. One possible explanation is a Peso problem: crashes (for which the payoff of a put is high) are taken into account for pricing, but are under-represented in the data sets used for empirical tests. This explanation is rejected by Bondarenko (2003b) who shows that his newly derived pricing restriction controlling for the peso problem is violated.

In this paper, we argue that the approach presented by Bondarenko (2003b) only solves the problem of missing terminal stock prices ('state peso problem'), but that the problem of missing paths for the price of the underlying ('path peso problem') remains. We derive analytical expressions for the effect of the path peso problem on the new pricing restriction, and we show under which conditions a correctly priced claim appears overpriced or underpriced in an empirical test. The potential magnitude of the path peso problem is analyzed in a simulation study. We argue that the results of existing empirical studies can be explained by the path peso problem, so that the high prices of put options do not necessarily reject standard asset pricing models.

JEL: G12, G13

Keywords: Put options, Peso problem, martingale restriction, pricing kernel, crash risk

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1 Introduction and Motivation

By now, it can be considered a stylized fact about the major option markets around the world that out-of-the-money (OTM) put options seem to be overpriced relative to standard option pricing models. There is no doubt that put options are quite risky investments and should thus earn significant risk premia. Since the return on a put is negatively correlated with the return on the underlying asset, a negative risk premium or, equivalently, a high price for this option is also in line with intuition. However, the absolute amount of the excess returns on puts is so large that it could be explained only by rather implausible parameter values and risk premia.

The related literature exhibits two main strands. First, there are studies focusing on the pure pricing aspect. Models with stochastic volatility and jumps are employed to explain the large negative slope of the implied volatility smile. However, the parameter values obtained from a calibration of these models to cross sections of option prices often appear implausible, and they are in most cases not in line with time series estimates, as shown for example by Bakshi, Cao, and Chen (1997), or by Bates (2000). One example for such a result is that the 'volatility of volatility' in stochastic volatility models often has to be very high to explain observed option prices. Second, there are studies investigating the time series properties of option prices, e.g. by computing expected option returns. The main result of these papers is that expected excess option returns and in particular the risk premia in put option prices cannot be explained by standard asset pricing models. Representatives of this type of analyzes are Coval and Shumway (2001), Bondarenko (2003b,c), or Bollen and Whaley (2004). The special properties of put option returns make a short sale of these derivative instruments seem like a very attractive investment. The advice is therefore that investors should hold significantly negative positions in these contracts, as suggested, among others, by Driessen and Maenhout (2003).

In this paper, we argue that the peso problem may explain the seemingly too high prices of put options. The term 'peso problem' describes the situation that rare events are anticipated when assets are priced, but do not occur often enough in the sample to justify the observed prices. The empirical distribution thus differs from the true distribution, and the empirical test is based on the wrong probability measure. For example, crashes with large negative price movements in the underlying have a large impact on put prices. In a situation where crashes are observed too infrequently in the sample, put prices appear too high, and consequently investors are advised to short puts as in Driessen and Maenhout (2003).

The studies closest to ours in the existing literature are the two papers by Bondarenko (2003b,c). He derives a new constraint on the price processes of traded assets, which is similar in spirit to the classical restriction from option pricing theory. There, under the risk-neutral measure, normalized prices must be martingales. The new restriction says that prices normalized by the risk-neutral density must be martingales under the objective measure when we additionally condition on the terminal value of the state variable.

While the classical martingale restriction is based on the assumption of no arbitrage, the economic interpretation of the new restriction is that, if it holds, there are no *statistical* arbitrage opportunities, or, alternatively, that the pricing kernel (or its projection) is path-independent. There are several important advantages to this innovative methodology. First, there is no need to specify a functional form for the pricing kernel (thus avoiding the classical joint hypotheses problem), since the restriction holds for a wide class of pricing models. Second, the restriction remains valid even under biased beliefs, i.e. in the case when the investor uses a wrong probability measure to price assets. And third, it also holds if there is a selection bias or a peso problem, meaning that some terminal states are missing in the sample. In an empirical study Bondarenko (2003c) tests the new restriction empirically for put options on the S&P 500. The result is a clear rejection, and puts again seem to be grossly overpriced.

Bondarenko (2003b) controls for the special case when the peso problem is relevant for terminal states. In our paper we extend the notion of a peso problem from the special case to the general one related to complete paths of the underlying state variables, that is to a 'path peso problem'. We thus consider the case when, given the true set of possible paths and their probabilities, extreme paths like those with very high volatility are missing in the empirical sample. In our analysis, we deal both with the well-known classical martingale restriction and the restriction derived by Bondarenko (2003b). First, we derive analytical expressions for the impact of the general path peso problem on both restrictions. The expressions show that the impact of peso problems on the two restrictions is structurally similar, and they show that the restriction of Bondarenko (2003b) is robust only with respect to the special case of a peso problem for terminal states. Second, we analyze in which direction (overpricing versus underpricing) the restrictions are violated in different scenarios. The direction depends on the joint characteristics of the rare paths or states and the contingent claim. If mainly paths (or states) with a high contribution to the initial price of the claim are missing, the claim will appear overpriced by the market (and vice versa).

Our theoretical arguments are illustrated within a very simple discrete setup. There, we also demonstrate that investment strategies implied by tests ignoring the path peso problem are successful whenever rare paths do not occur in the future, whereas they generate significant losses otherwise. We furthermore perform a simulation study to assess the impact of the peso problem. Given the empirical evidence discussed above, we focus on the pricing of out-of-the money (OTM) put options. The stock is assumed to follow a jump-diffusion process, and the pricing kernel is determined by a CRRA utility function. All assets are priced in a theoretically correct fashion, and both the classical martingale restriction and the restriction of Bondarenko (2003b) hold. If low terminal stock prices after a crash are missing in the sample, a test of the classical martingale restriction leads to the wrong conclusion that puts are overpriced, while the restriction derived by Bondarenko (2003b) still holds. If paths with a high realized variance are missing, then this restriction is violated, too. This shows that it can control for the peso problem for terminal states, but is not robust with respect to the general path peso problem.

Our main conclusion is that in the presence of a path peso problem puts are only seemingly overpriced. Taking into account that some paths of the underlying asset price occur too rarely in empirical samples compared to the true distribution, prices can still be explained by standard valuation models. Thus, based on these results, there is no need at the moment to resort to highly sophisticated option pricing models (which probably only increase estimation error) or to behavioral approaches.

The remainder of the paper is organized as follows. In Section 2 we present the general framework of our analysis. The restrictions imposed on option prices with and without the presence of peso problems are discussed in Section 3. The simulation study is presented in Section 4. Section 5 concludes.

2 Model Setup

2.1 Assumptions

Throughout this paper we assume that the observed empirical data are generated by some 'true' option pricing model which determines the distribution of the asset prices under the physical measure and the prices of contingent claims. Our objective is to analyze restrictions that hold in this pricing model and to show what happens to these restrictions in case of peso problems.

For the sake of simplicity, we work within a model with discrete time and a finite discrete state space. However, the concepts can be applied to continuous models in an analogous way. Furthermore, we set the interest rate equal to zero, which can be interpreted as the use of normalized prices. The physical probability measure is denoted by P, and Q stands for its risk-neutral counterpart. The pricing kernel (or Radon-Nikodym-derivative, since the interest rate is zero) is denoted by ξ:

$$\xi_T = \frac{dQ}{dP}$$

with

$$\xi_t = E^P[\xi_T|\mathcal{F}_t],$$

where $\mathcal{F}_t$ represents the information available at time t. The price C_t at time t for a claim C, which does not pay dividends, is then given by

$$C_t \quad = \quad E^P\left[C_u \frac{\xi_u}{\xi_t} \;\middle|\; \mathcal{F}_t\right].$$

2.2 Peso Problem

The term 'peso problem' describes a situation in which rare events are not observed in the sample, but are taken into account by investors for pricing, since they know that, theoretically, these rare events have positive probability. More generally, a peso problem implies that the true probability distribution cannot be estimated with sufficient precision from the empirical sample, so that important characteristics of the underlying probability law are not captured by the estimated distribution. A prominent example for the peso problem is that the probability of a crash is potentially underestimated empirically.

In our formal analysis the probability distribution estimated from the sample is $\widetilde{P}$, which will in general be different from the true distribution P. We assume the following Radon-Nikodym derivative of $\widetilde{P}$ with respect to P:

$$\frac{d\widetilde{P}}{dP} \quad = \quad \frac{1_A}{P(A)}. \tag{1}$$

With Ω as the event space all paths in $\Omega \setminus A$ are called 'rare', since they are missing in the sample. The probabilities of all 'normal' paths in A are correct in that they coincide with the true probabilities conditional on A. Note that P and $\widetilde{P}$ are not equivalent. $\frac{d\widetilde{P}}{dP}$ exists, but $\frac{dP}{d\widetilde{P}}$ does not, since the null sets of the two measures do not coincide.

We consider two possible characterizations of the set A. First, there may be a *peso problem for terminal states*. This means that events are categorized as rare or normal depending only on S_T, the stock price at the terminal time T. As an example consider the case when all terminal stock prices that deviate by more than 40% from the current price are rare. Then the two sequences of prices $\{S_0 = 100, S_1 = 50, S_2 = 50\}$ and $\{S_0 = 100, S_1 = 70, S_2 = 50\}$ are rare, but $\{S_0 = 100, S_1 = 50, S_2 = 100\}$ is not. Second, there may be what we call a *path peso problem*. In this case, it can depend on the whole path whether an event is classified as normal or as rare. For example, we consider all paths as rare where the stock price changes by more than 40% over one period. Now the paths $\{S_0 = 100, S_1 = 50, S_2 = 50\}$ and $\{S_0 = 100, S_1 = 50, S_2 = 100\}$ would be classified as rare, while $\{S_0 = 100, S_1 = 70, S_2 = 50\}$ would still be normal.

3 Restrictions on Option Prices

3.1 No Arbitrage: The Classical Martingale Restriction

The classical martingale restriction for the price of a derivative asset C is

$$C_t \xi_t \quad = \quad E^P \left[C_s \xi_s | \mathcal{F}_t \right] \qquad t \leq s \tag{2}$$

with ξ as the pricing kernel and P as the true probability measure. We assume that the classical martingale restriction holds in the true model, i.e. the economy is arbitrage-free. In this paper we rather focus on *empirical* phenomena that could lead to a rejection of the classical martingale restriction, despite theoretically correct pricing under the true model.

In case of a violation of Equation (2), e.g. when a claim is overpriced by the market, the arbitrage opportunity would consist of selling the claim and buying its replicating portfolio (so we implicitly assume that the claim is attainable). To interpret the violation from an asset pricing point of view, note that in this case the pricing kernel is proportional to marginal utility. An overpricing of the claim then implies that a short position in C_t (and an investment of the proceeds into the money market account) increases expected utility. The investor would still want to trade, the current prices of the claims would not be equilibrium prices, and the asset pricing model would be rejected.

For a test of the martingale restriction one has to specify some model for the pricing kernel. The probability distribution would be estimated from the time series, and this estimate is denoted by $\widetilde{P}$. The model price for asset C is then given by

$$E^{\widetilde{P}}\left[\frac{C_s \xi_s}{\xi_t} \,\middle|\, \mathcal{F}_t\right] \tag{3}$$

and it is compared to the observed market price. Even if we use the correct model to determine ξ and if the claims are priced correctly by the market, the model price calculated in Equation (3) can deviate from the value C_t in Equation (2) for several reasons which are e.g. discussed in Bondarenko (2003b). In the following, we focus on the peso problem. Rare events are missing in the sample, so that the empirical distribution $\widetilde{P}$ is different from the true physical distribution P. This may result in a deviation of the model price calculated in (3) from the true price C_t. We are thus facing the well-known problem of a test of joint hypotheses. We cannot tell if the martingale restriction is rejected, because a wrong model is used or because the empirical distribution deviates from the true distribution.

3.2 Peso Problem for the Classical Martingale Restriction

We now focus on the peso problem and its impact on the classical martingale restriction. The numerical examples in the following are based on a simple two-period binomial tree shown in Figure 1. Despite its simplicity, the model is rich enough to show the main arguments. The parameters of the model are $S_0 = 100$, $u = 1.2$, $d = 0.8$, and $r = 0$, where u is the jump parameter for an upward move of the stock, and d is the analogous number for a price decline. The risk-neutral probability of an up-move is thus equal to 1/2. For the physical probability measure, we assume that the probability of an up-move is 2/3. From these data the following pricing kernel results:

$$\xi_2^{(uu)} = \frac{9}{16}, \qquad \xi_2^{(ud)} = \xi_2^{(du)} = \frac{9}{8}, \qquad \xi_2^{(dd)} = \frac{9}{4},$$

where $\xi_2^{(uu)}$ is the pricing kernel at time $t = 2$ when the stock price has moved up twice, and the events ud, du, and dd are characterized analogously. The pricing kernel is the larger the smaller the stock price, which is in line with a setup where a risk averse agent holds the stock, and where his pricing kernel is proportional to his marginal utility. The claim C we want to price is a European put with maturity date $T = 2$ and a strike price equal to 100. A quick calculation shows that the price of this claim at time $t = 0$ is 11. The classical martingale restriction is trivially true, since we assume an arbitrage-free model:

$$\begin{aligned} C_0\xi_0 &= P(uu)C_2^{uu}\xi_2^{uu} + P(ud)C_2^{ud}\xi_2^{ud} + P(du)C_2^{du}\xi_2^{du} + P(dd)C_2^{dd}\xi_2^{dd} \\ \Leftrightarrow\ 11 \cdot 1 &= \frac{4}{9} \cdot 0 \cdot \frac{9}{16} + \frac{2}{9} \cdot 4 \cdot \frac{9}{8} + \frac{2}{9} \cdot 4 \cdot \frac{9}{8} + \frac{1}{9} \cdot 36 \cdot \frac{9}{4}. \end{aligned}$$

Here, $P(ij)$ stands for the physical probability of the event ij for $i, j \in \{u, d\}$.

Now assume there is a peso problem in that the lowest stock price at time $T = 2$, i.e. $S_2 = 64$, is missing in the sample. The empirical probability distribution for the four paths would then be

$$\begin{aligned}\widetilde{P}(uu) &= \frac{4/9}{8/9} = \frac{1}{2}\\ \widetilde{P}(ud) &= \frac{2/9}{8/9} = \frac{1}{4}\\ \widetilde{P}(du) &= \frac{2/9}{8/9} = \frac{1}{4}\\ \widetilde{P}(dd) &= \frac{0}{8/9} = 0.\end{aligned}$$

The model price of the put option follows from Equation (3):

$$\begin{aligned}E^{\widetilde{P}}\left[\frac{C_2\xi_2}{\xi_0}\right] &= \frac{1}{2}\cdot 0\cdot\frac{9}{16}+\frac{1}{4}\cdot 4\cdot\frac{9}{8}+\frac{1}{4}\cdot 4\cdot\frac{9}{8}+0\cdot 36\cdot\frac{9}{4}\\ &= 2.25.\end{aligned}$$

Note that we use the correct pricing kernel determined within the correct model, but the price is computed based on the 'wrong' probability distribution. The obvious conclusion is that the option is overpriced, and that the investor should sell it short to make a profit.

We now analyze the problem formally. Our first proposition explicitly characterizes the impact of the peso problem on the classical martingale restriction.

Proposition 1 (Classical Martingale Restriction: Peso Problem) *The true measure is P, the empirical probability distribution is $\widetilde{P}$. Any deviation between P and $\widetilde{P}$ is due to the peso problem. In all states at time t, the probability of ending up in a normal event at time T is non-negative, that is $E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t\right] > 0$. Then, the model price of the claim is given by*

$$E^{\widetilde{P}}\left[\frac{C_T\xi_T}{\xi_t} \;\Big|\; \mathcal{F}_t\right] = C_t\; E^P\left[\frac{C_T\xi_T}{C_t\xi_t}\cdot\frac{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_T\right]}{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t\right]} \;\Big|\; \mathcal{F}_t\right] \qquad T \geq t. \tag{4}$$

Proof: See Appendix A.1.

Note that the expectation on the left hand side is calculated under $\widetilde{P}$, whereas on the right hand side the true measure P is used. The price of the claim as calculated under $\widetilde{P}$ is equal to the true price times a factor due to the peso problem. Later on, we will see that the equation describing the impact of the path peso problem on the restriction of Bondarenko (2003b) has an identical structure.

For the simple case described by equation (1), the factor due to the peso problem is

$$E^P\left[\frac{C_T\xi_T}{C_t\xi_t}\cdot\frac{P(A|\mathcal{F}_T)}{P(A|\mathcal{F}_t)}\;\Big|\;\mathcal{F}_t\right], \tag{5}$$

as shown in Appendix A.3. By assumption, $P(A|\mathcal{F}_t) > 0$ so that as seen from time t, we may still end up in a normal event. If this condition were not met, the current state at time t would belong to a rare event for sure and would thus not be observed in the sample, so we would never perform the empirical test conditional on this state.

The peso problem factor (5) depends on a ratio of probabilities. For every event in $\mathcal{F}_T$, we consider the path that leads to this event and compare the probability of being in the set A of normal events at time T (which is either zero or one) to the probability as seen at time t (which is less than or equal to one). The fraction is thus equal to zero for rare events, greater than one for a normal event (when we are not sure at time t whether we will end up in A at time T) or equal to one (if we already know at time t that we are in a state from the set A). So intuitively probability mass is shifted from rare to normal events.

The classical martingale restriction is obviously violated if the peso problem factor (5) is different from one. It can be rewritten as

$$E^P\left[\frac{C_T\xi_T}{C_t\xi_t}\cdot\frac{P(A|\mathcal{F}_T)}{P(A|\mathcal{F}_t)}\;\Big|\;\mathcal{F}_t\right] \;=\; 1+cov^P\left[\frac{C_T\xi_T}{C_t\xi_t},\frac{P(A|\mathcal{F}_T)}{P(A|\mathcal{F}_t)}\;\Big|\;\mathcal{F}_t\right],$$

and it differs from one if the covariance between $C_T\xi_T$ and the ratio of the two probabilities is not zero.

For a negative covariance the factor is less than one, and the claim appears overpriced. The intuition behind this case is that $C_T\xi_T$ is large for rare states (where the ratio of probabilities is zero) and small for normal states (where the ratio of probabilities is at least one), so that states with relatively large contributions to the value of the claim are missing in the empirical sample. In case the investor implements the (seemingly existing) arbitrage strategy and ignores the peso problem, a gain will be generated as long as only normal states are realized, with potentially large losses otherwise. For a positive covariance the argument is completely analogous, resulting in seemingly undervalued claims.

Note, however, that there are special cases under which the classical martingale restriction still holds even in presence of a peso problem. The ratio of probabilities in Equation (4) is, of course, identically equal to one if we already know at time t that we are in a normal event. This is the limiting case where there is no real peso problem, and where the classical martingale restriction holds for all claims. If the covariance between the ratio of probabilities and $C_T\xi_T$ is zero, we have a case where a peso problem is present, but it does not affect the price of C in a systematic way. $C_T\xi_T$ is neither systematically higher nor systematically lower for rare events, and errors just offset each other, i.e. we 'forget' states with both high *and* low contributions to the claim price. The classical martingale restriction then still holds for claims that meet this 'zero covariance restriction'.

As an example consider again an OTM put on the stock. Assume that rare events are those with very low terminal stock prices (crashes). After a crash, the put price is high (the option may now even be in the money) and the pricing kernel is high (low aggregate wealth implies large marginal utility), so that $C_T\xi_T$ is large for a rare event, for which the factor due to the Peso problem is smaller than one. This implies that the put seems overpriced. The arbitrage strategy in this case would be to sell the put and buy a portfolio of Arrow-Debreu securities, neglecting rare terminal states. The difference between the market and the model price is consumed immediately. If a normal event is realised, the payoff from the strategy is zero. Otherwise a loss will emerge.

3.3 No Statistical Arbitrage: The Restriction of Bondarenko (2003b)

Since our work represents a generalisation of the approach suggested by Bondarenko (2003b), we will now briefly review his results. The restriction he derives controls for the peso problem for terminal states:

Proposition 2 (Bondarenko (2003b)) *The true measure is P, the empirical probability distribution is $\widetilde{P}$. Any deviation between P and $\widetilde{P}$ is due to the peso problem. The pricing kernel is path-independent, i.e $\xi_T = \xi(S_T)$. The terminal stock price x is chosen such that $Q(S_T = x|\mathcal{F}_u) > 0$ for all $\mathcal{F}_u$. Conditional on the terminal stock price x the probability measures P and $\widetilde{P}$ coincide. Then the following restriction holds under $\widetilde{P}$*

$$E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \,\Big|\, \mathcal{F}_t, S_T = x\right] = \frac{C_t}{Q(S_T = x|\mathcal{F}_t)} \qquad u \geq t. \tag{6}$$

Proof: See Bondarenko (2003b).

As we will show below this result represents a special case of our Proposition 3.

The main point to note is that in Equation (6) everything is conditioned on the observed terminal stock price. Therefore, terminal values missing from the sample do not matter, and as long as conditional probabilities coincide under P and under $\widetilde{P}$, the restriction holds for both probability measures. Bondarenko (2003b) assumes conditional probabilities to be correct, which he associates with a learning process on the part of investors. We will later on argue that it is exactly this assumption of correct conditional probabilities, which will be violated if there is a path peso problem.

Furthermore, Proposition 2 resolves the problem of a test of joint hypotheses. There is no need to pre-specify a pricing kernel, but the restriction has to hold for all models with a path-independent pricing kernel. This means that the restriction can be tested for a whole class of models simultaneously. The risk-neutral probabilities in Equation (6) can be estimated from cross-sections of option prices using standard methods, see for example Rubinstein (1994), Jackwerth (1999), or Bondarenko (2003a). Bondarenko (2003c) shows in detail how to test the restriction (6) empirically, and we will also discuss this issue in Section 4.

The economic intuition behind the result is that it is equivalent to the absence of statistical arbitrage opportunities, as shown in Bondarenko (2003b). A statistical arbitrage opportunity is a portfolio with zero price today and payoff Y_T at time T such that

$$\begin{aligned} E^P\left[Y_T|\mathcal{F}_t, S_T = x\right] &\geq 0 \qquad \forall\, x \\ E^P\left[Y_T|\mathcal{F}_t\right] &> 0. \end{aligned}$$

An investor whose 'optimal' portfolio has a payoff at time T which only depends on the terminal stock price S_T can increase his expected utility by buying Y. Consequently, there can be no equilibrium in the presence of statistical arbitrage opportunities. Note that the exclusion of statistical arbitrage opportunities does not put an upper bound on the Sharpe ratio or the profitability of portfolios. The restriction thus differs from approaches ruling out investment opportunities that are 'too good', like Cochrane and Saa-Requejo (2000) and Cerný (2003).

We will now reconsider the numerical example from Section 3.1 and assume that there is the same peso problem for terminal states leading to a violation of the classical martingale restriction. The model price of the claim is now

$$
\begin{aligned}
&Q(S_T = x|\mathcal{F}_t)E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \,\Big|\, \mathcal{F}_t, S_T = x\right] \\
&= Q(S_2 = 96|\mathcal{F}_0)E^{\widetilde{P}}\left[\frac{C_1}{Q(S_2 = 96|\mathcal{F}_1)} \,\Big|\, \mathcal{F}_0, S_2 = 96\right] \\
&= \frac{2}{4}\left\{\frac{1/4}{2/4}\cdot\frac{2}{1/2} + \frac{1/4}{2/4}\cdot\frac{20}{1/2}\right\} \\
&= 11.
\end{aligned}
$$

The restriction of Bondarenko (2003b) thus holds, and despite the peso problem for terminal states, our simple asset pricing model would no longer be rejected by the data.

3.4 Path Peso Problem

As seen above, the approach suggested by Bondarenko (2003b) controls for the peso problem, at least in the case when this problem is relevant for terminal states. A key assumption here is that, while unconditional probabilities of terminal states are subject to mis-estimation, the conditional probabilities of paths that lead to normal terminal states are not. Put differently, we assume to observe all paths leading to normal terminal states with the correct probability.

This is exactly the point we are going to address in the following. We claim that the approach cannot solve the more general issue of mis-estimation of *path* probabilities. This means the restriction in Equation (6) will not hold if the conditional path probabilities cannot be estimated precisely. Rare paths leading to normal terminal states are thus missing in the sample, and this is what we call the 'path peso problem'. Given a terminal state and the set of all paths that end in this state, some of these paths are missing in the sample. The problem resulting from this incomplete observation of the model is that some path probabilities in the empirical distribution are zero, and if these paths contribute to the initial price of a claim significantly, the claim will seem either overpriced or underpriced in the sample.

Consider again our numerical example. Assume now that the path du, for which the stock price first drops and then goes up again, is missing in the sample. The empirical probability distribution for paths will then be

$$\begin{aligned}
\widetilde{P}(uu) &= \frac{4/9}{7/9} = \frac{4}{7} \\
\widetilde{P}(ud) &= \frac{2/9}{7/9} = \frac{2}{7} \\
\widetilde{P}(du) &= \frac{0}{7/9} = 0 \\
\widetilde{P}(dd) &= \frac{1/9}{7/9} = \frac{1}{7},
\end{aligned}$$

and the conditional path probabilities used in the test are

$$\begin{aligned}
\widetilde{P}(S_1 = 120|S_2 = 96) &= 1 \\
\widetilde{P}(S_1 = 80|S_2 = 96) &= 0.
\end{aligned}$$

This gives the following model price of the claim in the context of Bondarenko (2003b)'s restriction:

$$\begin{aligned}
&Q(S_T = x|\mathcal{F}_t)E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \,\Big|\, \mathcal{F}_t, S_T = x\right] \\
&= Q(S_2 = 96|\mathcal{F}_0)E^{\widetilde{P}}\left[\frac{C_1}{Q(S_2 = 96|\mathcal{F}_1)} \,\Big|\, \mathcal{F}_0, S_2 = 96\right] \\
&= \frac{2}{4}\left\{1 \cdot \frac{2}{1/2} + 0 \cdot \frac{20}{1/2}\right\} \\
&= 2.
\end{aligned}$$

The restriction from Proposition 2 is violated, and the put seems overpriced. The statistical arbitrage strategy is to sell the put for 11, consume 9 immediately, and use the remaining 2 to buy Arrow-Debreu securities for the state $S_2 = 96$. At time 1, the portfolio is sold, and all the proceeds are invested into Arrow-Debreu securities for the state $S_2 = 96$. The expected payoff of this strategy in state $S_2 = 96$ is zero under the empirical measure $\widetilde{P}$, but is negative under the true measure. The utility of the investor thus increases if a normal path is realized, but will be lower in the other case.

We will now discuss the general form of the peso problem in detail and analyze its implications for the restriction (6). Our main result is contained in the next proposition.

Proposition 3 (Path Peso Problem) *The true measure is P, the empirical probability distribution is $\widetilde{P}$. Any deviation between P and $\widetilde{P}$ is due to the path peso problem. The pricing kernel ξ is path-independent, and the terminal stock price x is chosen such that*

$Q(S_T = x|\mathcal{F}_u) > 0$ for all $\mathcal{F}_u$. Furthermore, $E^P\left[\frac{d\tilde{P}}{dP} \mid \mathcal{F}_t, S_T = x\right] > 0$. The model price of the claim is then given by

$$
\begin{aligned}
&Q(S_T = x|\mathcal{F}_t)E^{\tilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \mid \mathcal{F}_t, S_T = x\right] \\
&= \; C_t \; E^P\left[\frac{C_u\xi_u}{C_t\xi_t} \frac{E^P\left[\frac{d\tilde{P}}{dP} \mid \mathcal{F}_u, S_T = x\right]}{E^P\left[\frac{d\tilde{P}}{dP} \mid \mathcal{F}_t, S_T = x\right]} \mid \mathcal{F}_t\right] \qquad u \geq t.
\end{aligned}
$$

Proof: See Appendix A.2.

As shown in Appendix A.4, for the simple case given in Equation (1) where all paths from $\Omega \setminus A$ are omitted, the restriction becomes

$$
\begin{aligned}
&Q(S_T = x|\mathcal{F}_t)E^{\tilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \mid \mathcal{F}_t, S_T = x\right] \\
&= \; C_t \; E^P\left[\frac{C_u\xi_u}{C_t\xi_t} \cdot \frac{P(A|\mathcal{F}_u, S_T = x)}{P(A|\mathcal{F}_t, S_T = x)} \mid \mathcal{F}_t\right]. \qquad (7)
\end{aligned}
$$

The correction factor for the path peso problem is thus given by

$$
E^P\left[\frac{C_u\xi_u}{C_t\xi_t} \cdot \frac{P(A|\mathcal{F}_u, S_T = x)}{P(A|\mathcal{F}_t, S_T = x)} \mid \mathcal{F}_t\right], \qquad (8)
$$

which depends on the ratio of two conditional probabilities. To interpret this ratio, consider a path which ends in $S_T = x$. At time t, we calculate the conditional probability that this path is in the set A of normal paths, given the observation of the path up to time t and given that we end up in $S_T = x$. At time u, we recompute this conditional probability. If the given path is now, due to the observation from t to u, more likely to belong to the set A than as seen from time t, the probability ratio in (8) is larger than one (and vice versa). In the limiting case when we already know at time u that the observation from time t to time u belongs to a rare path, the ratio is zero. Similar to the intuition for the state peso problem, probability is shifted from rare to normal paths in the estimation of the empirical distribution.

First, we now want discuss a special case in which Proposition 3 simplifies to the restriction of Bondarenko (2003b). Assume that there is a peso problem related to terminal states only. Then, a rare path is by definition one that ends in a rare terminal state, i.e.

$$
A = \{\text{normal paths}\} = \{\text{normal terminal states } S_T\}.
$$

We assume that $S_T = x$ is a normal terminal state, since otherwise an empirical test conditional on $S_T = x$ would be meaningless. Then the conditioning information that the path ends in $S_T = x$ already tells us that we are on a normal path, so that

$$P(A|\mathcal{F}_u, S_T = x) = P(A|\mathcal{F}_t, S_T = x) = 1,$$

and the ratio of probabilities in (8) is identically equal to one. The peso problem factor becomes

$$E^P\left[\frac{C_u\xi_u}{C_t\xi_t}\cdot\frac{P(A|\mathcal{F}_u, S_T = x)}{P(A|\mathcal{F}_t, S_T = x)}\;\Big|\;\mathcal{F}_t\right] \;=\; E^P\left[\frac{C_u\xi_u}{C_t\xi_t}\;\Big|\;\mathcal{F}_t\right] \;=\; 1,$$

where we have used that the classical martingale restriction holds under the physical measure P. In this case, Proposition 3 thus simplifies to the restriction derived by Bondarenko (2003b).

Now consider the general form of the path peso problem. Assume that the factor in (8) is less than one. In this case, the claims appears overpriced by the market. The intuition is that $C_u\xi_u$ is large on rare paths and small on normal paths, so that paths with a large contribution to the initial price are missing in the sample. As an example consider the familiar deep OTM put. A rare path could be one on which a crash between t and u is followed by an increase of the stock price back to the initial value x. After the crash the put price at time u is high due to the low stock price, and the pricing kernel at time u is also high, since current wealth is low. These rare paths contribute significantly to the price of the put at time u, so the factor in (8) is less than one, and the put seems overpriced.

An investor implementing a statistical arbitrage strategy and ignoring the path peso problem will have a higher utility on normal paths, accompanied by a lower utility on rare paths. In case of the OTM put, he will thus loose money if the stock price falls and recovers afterwards. This problem is economically highly relevant. A short seller of such a put option is not only concerned about the terminal payoff of his position, but also about the value before maturity, since in the extreme case one single bad day might force him to leave the market.

4 Simulation Study

Our results show that an empirically documented overpricing of put options by the market may be explained by a peso problem for terminal states in case the classical martingale restriction is rejected, and by a path peso problem in case the restriction of Bondarenko

(2003b) is rejected. While it is thus possible to explain the direction of the mispricing, the question remains whether the observed size of the mispricing can also be explained.

To address this point, we perform a simulation study. We use a model setup where both the classical restriction and the restriction of Bondarenko (2003b) hold. Then we test the restrictions in the full sample and also when a certain number of paths are missing. The omission of paths is either caused by a peso problem for terminal states or by a path peso problem. As a result, we obtain a relation between the size of the mis-pricing and the number and characteristics of missing paths. The smaller the number of missing paths necessary to generate a certain amount of mis-pricing, the more likely this mis-pricing can have been caused by a path peso problem. In an empirical study, a similar idea could be used. There, one would try to find a 'true' distribution for which both the classical restriction and the restriction of Bondarenko (2003b) hold and which results in the empirical distribution if there is a peso problem. One would then again judge by by the number and properties of the eliminated paths whether it seems plausible or not that these paths are indeed missing in the sample.

We assume that the true data-generating model is the jump-diffusion model of Merton (1976). As a benchmark, we will later on also consider the model of Black and Scholes (1973). Both models allow for a path-independent pricing kernel, so that the assumption of Bondarenko (2003b) is fulfilled. Under the true physical measure P, the stochastic differential equation for the stock price in the jump-diffusion model is

$$dS_t \;=\; \left(\mu - E^P[e^X - 1]h^P\right) S_t dt + \sigma S_t dW_t + S_{t-}\left(e^{X_t} - 1\right) dN_t,$$

where W is a Wiener process, N is a Poisson process with intensity h^P, and X is the jump size of log returns with $X \sim N\big(\ln(1+\mu_X^P) - 0.5\sigma_X^2, \sigma_X\big)$. For the sake of simplicity, we again assume that the interest rate is equal to zero.

The market is incomplete, and to determine the risk neutral measure Q, we assume that there is a representative investor with a power utility function. His constant relative risk aversion is denoted by γ. The pricing kernel ξ is then given by

$$\begin{aligned}\xi_t &= \frac{S_t^{-\gamma}}{E^P\left[S_t^{-\gamma}\right]} \\ &= \exp\left\{-\sigma\gamma W_t - 0.5\gamma^2\sigma^2 t\right\} \cdot \exp\left\{-\int_0^t \gamma X_u dN_u - E^P\left[e^{-\gamma X_u} - 1\right] h^P t\right\}.\end{aligned}$$

It is a function of the stock price only and thus path-independent. The resulting stock price process under the risk-neutral measure Q is

$$dS_t \;=\; -E^Q[e^X - 1]h^Q S_t dt + \sigma S_t d\widehat{W}_t + S_{t-}\left(e^{X_t} - 1\right) dN_t$$

where $\widehat{W}$ is a Wiener process, N is a Poisson process with intensity h^Q, and X is the jump size of log returns with $X \sim N\big(\ln(1+\mu_X^Q) - 0.5\sigma_X^2, \sigma_X\big)$. The relation between the true measure and the risk-neutral measure follows from the pricing kernel, and we get

$$\begin{aligned} d\widehat{W}_t &= dW_t + \gamma\sigma dt \\ h^Q &= h^P E^P\left[e^{-\gamma X_u}\right] \\ &= h^P e^{-\gamma \ln(1+\mu_X^P)+0.5\gamma(1+\gamma)\sigma_X^2} \\ \mu_X^Q &= (1+\mu_X^P)e^{-\gamma\sigma_X^2} - 1. \end{aligned}$$

Since the expected rate of return of the stock under the risk-neutral measure Q has to be equal to the short rate which is zero in our setup, we can furthermore conclude that the expected excess return μ under the true measure is

$$\mu = \gamma\sigma^2 + \mu_X^P h^P - \mu_X^Q h^Q.$$

The restrictions are formulated in terms of excess returns. The classical martingale restriction then reads

$$E^P\left[\frac{C_u - C_t}{C_t} \cdot \frac{\xi_u}{\xi_t} \;\Big|\; \mathcal{F}_t\right] = 0, \tag{9}$$

and the restriction of Bondarenko (2003b) becomes

$$E^P\left[\frac{C_u - C_t}{C_t} \cdot \frac{Q(S_T|\mathcal{F}_t)}{Q(S_T|\mathcal{F}_u)} \;\Big|\; \mathcal{F}_t\right] = 0 \tag{10}$$

where we have taken the expectation over all terminal stock prices. The restrictions hold under the true measure P, and we will analyze in the following if and in which direction these restrictions are violated when there is a peso problem.

Since there is no closed form solution for the expectations in (9) and (10) under the empirical measure $\widetilde{P}$, we perform a Monte-Carlo simulation with 1,000,000 paths. This large number of runs was chosen to achieve a good fit of the empirical to the true distribution. We use $S_0 = 100$, $r = 0$, $\sigma = 0.15$, $h^P = 0.5$, $\mu_X^P = -0.10$, $\sigma_X = 0.15$, and a relative risk aversion of $\gamma = 3$. To ensure that the distribution in the sample is sufficiently close to the true distribution, the distribution of the jump sizes in the sample has to be close to the true one. For the time intervals, we therefore set $t = 0$, $u = 0.5$, and $T = 1$, which can be seen as an analysis under a magnifying glass, compared to the setup of Bondarenko (2003b) who considers intervals of one month each. The test assets are OTM puts with a maturity of one year. To assess the impact of the peso problem, we omit a certain number of paths from the sample, and then test the restrictions in this reduced sample.

For both restrictions (9) and (10), we calculate the sample means in the original sample and in the samples subject to peso problems. We can interpret the sample mean of these statistics as the percentage overpricing of the claim by the model (a negative value corresponds to the claim being overpriced by the market), and we then use a simple t-test to see whether these percentage pricing errors are significantly different from zero. We do not claim the t-test to be the best choice here, but it turns out that if the null hypotheses of zero pricing errors is rejected, the t-statistics are very high (in many cases in absolute terms well above 100), so that the mean pricing errors are very different from zero in terms of standard deviations and the results are robust.

When there is a peso problem for terminal states, we omit the paths with the largest absolute returns in excess of μ from t to T, i.e. with the largest values of $\left|\frac{\ln S_T - \ln S_t}{T-t} - \mu\right|$. Given that large returns in absolute terms are primarily caused by jumps and given that we assume a negative mean jump size, this implies that we mainly drop paths with very low terminal stock prices.

The classical martingale restriction will tend to signal negative pricing errors and an overpricing of puts by the market. Figure 2 gives the mean percentage pricing errors for OTM puts and the t-statistics as a function of the share $P(\Omega \setminus A)$ of missing paths. If 1% of the paths are missing, the percentage pricing errors range from -30% for a put that is 4%-OTM up to -60% for a put that is 20%-OTM. These pricing errors are highly significant. Even if only 0.05% of the paths (that is one out of every 2,000) are missing, the pricing errors range from -13% to -7%.

The restriction of Bondarenko (2003c) still holds. Figure 3 gives the mean percentage pricing errors for OTM puts and the associated t-statistics, again as a function of the probability of a path missing from the sample. The pricing errors are not significantly different from zero, and pricing models with a path-independent pricing kernel cannot be rejected. If the overpricing of put options by the market signaled by the classical martingale restriction is only caused by a peso problem for terminal states, a test of the new restriction thus avoids the improper rejection of the pricing model.

To represent a path peso problem, we omit the paths with the largest sum of absolute returns in excess of μ over the two subperiods $[t, u]$ and $[u, T]$, i.e. with the largest values for the variable $y = \left|\frac{\ln S_u - \ln S_t}{u-t} - \mu\right| + \left|\frac{\ln S_T - \ln S_u}{T-u} - \mu\right|$. This can be interpreted in some sense as omitting the paths with the largest variation.

The classical martingale restriction is again rejected, and puts seem overpriced. Figure 4 gives the mean percentage pricing errors for OTM puts and the t-statistics as a function of the probability $P(\Omega \setminus A)$ of missing paths. Both the pricing errors and the t-statistics are comparable to the results in case of the peso problem for terminal states.

The restriction of Bondarenko (2003b), which is robust with respect to the peso problem for terminal states, is now also violated. The mean percentage pricing errors for OTM puts and the t-statistics (again as a function of the probability $P(\Omega \setminus A)$ of missing paths) are given in Figure 5. Compared to the results for the classical martingale restriction, the mis-pricing is reduced. However, if 1% of the paths are missing, the percentage pricing errors are still between -25% and -40%, and they are highly significant. Our simulation furthermore shows that the restriction is very sensitive to influential paths. From an inspection of the sample, we see that for these influential paths, the stock price first drops until time u and then recovers to at least the initial value. In this case, the price of the put and the pricing kernel at time u are both large, so that the contribution of these influential paths to the price of the put is large. At the same time, the variation of the stock price on these paths is high, so that according to our criterion they are likely to be rare paths missing from the sample.

The results of the simulation thus confirm the theoretical analysis from Section 3. Even if the tested model is correct, both the classical martingale restriction and the new restriction derived by Bondarenko (2003b) may be rejected empirically. We now check the robustness of these results.

First, we test whether we obtain the same results when the missing paths are chosen by chance rather than based on some criterion, like a high variation of the stock price. Figure 6 shows the results for the test of the classical martingale restriction and the restriction of Bondarenko (2003b). Even if up to 95% of the paths are missing from the original sample, both restrictions cannot be rejected. This confirms that the way in which the omitted paths are selected is indeed crucial. Only if the contribution of these paths to the price of a claim is systematically higher (lower) than for the paths still in the sample, the claim appears overpriced (underpriced) by the market.

Second, we test whether the results are driven by the choice of a jump-diffusion model. We thus perform an analogous simulation study for the model of Black and Scholes (1973). The results are given in Figure 7 for the case of a peso problem for terminal states and in Figure 8 for the case of a path peso problem. The criteria for the omission of terminal states/paths are the same as above. Again, the classical martingale restriction is rejected in both cases, while the restriction of Bondarenko (2003b) is only violated if there is a path peso problem. The inclusion of a jump component is thus not critical.

Third, we redo the Monte Carlo simulation with 240 instead of 1,000,000 paths to analyze if the results still hold for a sample size that is realistic for empirical studies. Figure 9 shows the results for the test of the classical martingale restriction, while Figure 10 gives the results for the test of the restriction of Bondarenko (2003b). The criteria for the omission of terminal states/paths are the same as above. The noise in the data is larger, but the conclusions do not change. A violation of the classical martingale restriction occurs if there is a peso problem, and the restriction of Bondarenko (2003b) holds in case of a peso problem for terminal states, but not if there is a peso problem for paths.

5 Conclusion

In this paper we discuss issues that arise in empirical tests of option pricing models. A rejection of the classical martingale restriction and a seeming mis-pricing of options can be explained by a peso problem, where the probability of rare events is underestimated in the sample. Assuming that some terminal stock prices are missing in the sample, Bondarenko (2003b) derives a new restriction that is not affected by this state peso problem. However, we show that even a rejection of this restriction does not imply that some given pricing model does not hold. An alternative explanation would be the existence of a path peso problem which represents a generalized version of the state peso problem. As the name says, the path peso problem relates to complete paths of the underlying, and the peso problem for terminal states can be shown to be a special case. Once rare paths are taken into account, seemingly mis-priced claims cease to be over- or undervalued, and seemingly good strategies cease to be (statistical) arbitrage opportunities.

So, despite many empirical studies pointing in the opposite direction, put option prices may still be in line with simple option pricing models. Further research could focus on an improvement of econometric techniques allowing the researcher to make better inferences about rare events or paths. Once there is a methodology which makes the path problem negligible, empirical studies could yield more reliable results concerning the validity of pricing models.

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A Appendix

A.1 Proof of Proposition 1

We take the expectation under $\widetilde{P}$ of the price multiplied by the pricing kernel and first change the measure from $\widetilde{P}$ to P:

$$E^{\widetilde{P}}\left[C_T\xi_T|\mathcal{F}_t\right] = E^P\left[\frac{\frac{d\widetilde{P}}{dP}}{E^P[\frac{d\widetilde{P}}{dP}|\mathcal{F}_t]}\,C_T\xi_T|\mathcal{F}_t\right].$$

The law of iterated expectations then gives

$$E^{\widetilde{P}}\left[C_T\xi_T|\mathcal{F}_t\right] = C_t\xi_t E^P\left[\frac{C_T\xi_T}{C_t\xi_t}\,\frac{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_T\right]}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_t\right]}\;\Big|\;\mathcal{F}_t\right]$$

A.2 Proof of Proposition 3

We take the expectation under $\widetilde{P}$ of the price normalized by the risk-neutral probability at time u for $S_T = x$ under the additional condition $S_T = x$ on the terminal value of the state variable. Multiplying with and dividing by this ratio at time t gives

$$\begin{aligned}
&E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T=x|\mathcal{F}_u)}\;\Big|\;\mathcal{F}_t, S_T=x\right]\\
&= \frac{C_t}{Q(S_T=x|\mathcal{F}_t)}\,E^{\widetilde{P}}\left[\frac{C_u}{C_t}\,\frac{Q(S_T=x|\mathcal{F}_t)}{Q(S_T=x|\mathcal{F}_u)}\;\Big|\;\mathcal{F}_t, S_T=x\right]\\
&= \frac{C_t}{Q(S_T=x|\mathcal{F}_t)}\,E^{\widetilde{P}}\left[\frac{C_u}{C_t}\,\frac{E^Q[1_{S_T=x}|\mathcal{F}_t]}{E^Q[1_{S_T=x}|\mathcal{F}_u]}\;\Big|\;\mathcal{F}_t, S_T=x\right].
\end{aligned}$$

Since the interest rate is assumed to be identically equal to zero, the pricing kernel ξ coincides with the Radon-Nikodym derivative of the risk-neutral probability measure Q with respect to the true probability measure P. We can thus rewrite the expressions $Q(S_T = x|\mathcal{F}_t)$ and $Q(S_T = x|\mathcal{F}_u)$ to obtain

$$\begin{aligned}
&E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T=x|\mathcal{F}_u)}\;\Big|\;\mathcal{F}_t, S_T=x\right]\\
&= \frac{C_t}{Q(S_T=x|\mathcal{F}_t)}\,E^{\widetilde{P}}\left[\frac{C_u}{C_t}\,\frac{E^P\left[1_{S_T=x}\frac{\xi_T}{\xi_t}\;\big|\;\mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\frac{\xi_T}{\xi_u}\;\big|\;\mathcal{F}_u\right]}\;\Big|\;\mathcal{F}_t, S_T=x\right]\\
&= \frac{C_t}{Q(S_T=x|\mathcal{F}_t)}\,E^{\widetilde{P}}\left[\frac{C_u\xi_u}{C_t\xi_t}\,\frac{E^P\left[1_{S_T=x}\xi_T\;\big|\;\mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\xi_T\;\big|\;\mathcal{F}_u\right]}\;\Big|\;\mathcal{F}_t, S_T=x\right].
\end{aligned}$$

In the next step, we use Bayes' theorem to get rid of the additional condition $S_T = x$ and then apply the tower law:

$$
\begin{aligned}
& E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \,\Big|\, \mathcal{F}_t, S_T = x\right] \\
&= \frac{C_t}{Q(S_T = x|\mathcal{F}_t)} E^{\widetilde{P}}\left[\frac{C_u\xi_u}{C_t\xi_t} \frac{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_u\right]} \frac{1_{S_T=x}}{E^{\widetilde{P}}\left[1_{S_T=x} \mid \mathcal{F}_t\right]} \,\Big|\, \mathcal{F}_t\right] \\
&= \frac{C_t}{Q(S_T = x|\mathcal{F}_t)} E^{\widetilde{P}}\left[\frac{C_u\xi_u}{C_t\xi_t} \frac{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_u\right]} \frac{E^{\widetilde{P}}\left[1_{S_T=x} \mid \mathcal{F}_u\right]}{E^{\widetilde{P}}\left[1_{S_T=x} \mid \mathcal{F}_t\right]} \,\Big|\, \mathcal{F}_t\right].
\end{aligned}
$$

The third factor within the new expectation is the ratio of two expectations which are calculated under $\widetilde{P}$. For these two expectations, we now use the Radon-Nikodym derivative of $\widetilde{P}$ with respect to P to write them as expectations under P:

$$
\begin{aligned}
& E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \,\Big|\, \mathcal{F}_t, S_T = x\right] \\
&= \frac{C_t}{Q(S_T = x|\mathcal{F}_t)} E^{\widetilde{P}}\left[\frac{C_u\xi_u}{C_t\xi_t} \frac{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_u\right]} \frac{E^P\left[1_{S_T=x}\frac{\frac{d\widetilde{P}}{dP}}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_u\right]} \mid \mathcal{F}_u\right]}{E^P\left[1_{S_T=x}\frac{\frac{d\widetilde{P}}{dP}}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_t\right]} \mid \mathcal{F}_t\right]} \,\Big|\, \mathcal{F}_t\right].
\end{aligned}
$$

Then we do the same for the outer expectation which is also calculated under $\widetilde{P}$ and simplify the resulting expression:

$$
\begin{aligned}
& E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \,\Big|\, \mathcal{F}_t, S_T = x\right] \\
&= \frac{C_t}{Q(S_T = x|\mathcal{F}_t)} E^{P}\left[\frac{C_u\xi_u}{C_t\xi_t} \frac{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_u\right]} \right. \\
&\qquad \left. \frac{E^P\left[1_{S_T=x}\frac{\frac{d\widetilde{P}}{dP}}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_u\right]} \mid \mathcal{F}_u\right]}{E^P\left[1_{S_T=x}\frac{\frac{d\widetilde{P}}{dP}}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_t\right]} \mid \mathcal{F}_t\right]} \frac{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_u\right]}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_t\right]} \,\Big|\, \mathcal{F}_t\right] \\
&= \frac{C_t}{Q(S_T = x|\mathcal{F}_t)} E^{P}\left[\frac{C_u\xi_u}{C_t\xi_t} \frac{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_t\right]}{E^P\left[1_{S_T=x}\xi_T \mid \mathcal{F}_u\right]} \frac{E^P\left[1_{S_T=x}\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_u\right]}{E^P\left[1_{S_T=x}\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t\right]} \,\Big|\, \mathcal{F}_t\right].
\end{aligned}
$$

For the inner expectation, we again condition on $S_T = x$. Several terms cancel, and we get

$$
\begin{aligned}
& E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \;\Big|\; \mathcal{F}_t, S_T = x\right] \\
= & \frac{C_t}{Q(S_T = x|\mathcal{F}_t)}\, E^P\left[\frac{C_u\xi_u}{C_t\xi_t}\, \frac{E^P\left[\xi_T \mid \mathcal{F}_t, S_T = x\right]}{E^P\left[\xi_T \mid \mathcal{F}_u, S_T = x\right]}\, \frac{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_u, S_T = x\right]}{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t, S_T = x\right]} \;\Big|\; \mathcal{F}_t\right].
\end{aligned}
$$

By assumption, the pricing kernel is path-independent, that is $\xi_T = \xi(S_T)$. This gives

$$
\begin{aligned}
& E^{\widetilde{P}}\left[\frac{C_u}{Q(S_T = x|\mathcal{F}_u)} \;\Big|\; \mathcal{F}_t, S_T = x\right] \\
= & \frac{C_t}{Q(S_T = x|\mathcal{F}_t)}\, E^P\left[\frac{C_u\xi_u}{C_t\xi_t}\, \frac{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_u, S_T = x\right]}{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t, S_T = x\right]} \;\Big|\; \mathcal{F}_t\right].
\end{aligned}
$$

A.3 Ratio of Probabilities in Proposition 1

The ratio of probabilities is

$$
\frac{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_T\right]}{E^P\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_t\right]}
$$

If the measure $\widetilde{P}$ is given by

$$
\frac{d\widetilde{P}}{dP} = \frac{1_A}{E^P[1_A]}
$$

then the ratio of probabilities becomes

$$
\frac{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_T\right]}{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t\right]} = \frac{P(A|\mathcal{F}_T)}{P(A|\mathcal{F}_t)}.
$$

A.4 Ratio of Probabilities in Proposition 3

The probability ratio captures deviations between P and $\widetilde{P}$

$$
\frac{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_u, S_T = x\right]}{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t, S_T = x\right]}.
$$

If the measure $\widetilde{P}$ is given by

$$\frac{d\widetilde{P}}{dP} = \frac{1_A}{E^P[1_A]},$$

then the ratio of probabilities becomes

$$\frac{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_u, S_T = x\right]}{E^P\left[\frac{d\widetilde{P}}{dP} \mid \mathcal{F}_t, S_T = x\right]} = \frac{P(A|\mathcal{F}_u, S_T = x)}{P(A|\mathcal{F}_t, S_T = x)}.$$

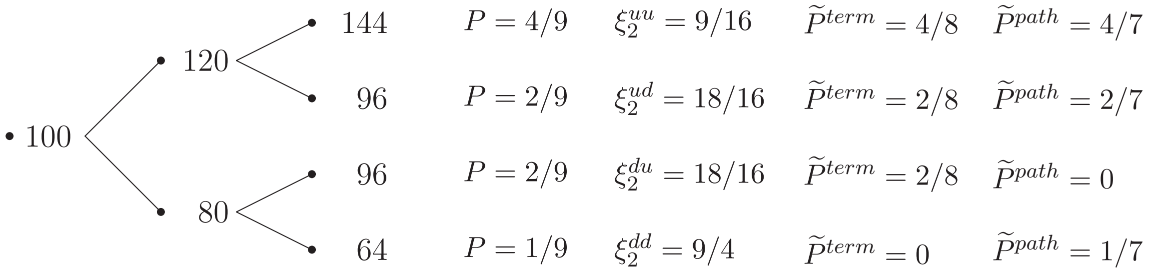

Figure 1: Binomial model

The parameters of the tree are $u = 1.2, d = 0.8, r = 0$. The true probability P of an up-move is 2/3, and together with the risk-neutral probability of 1/2, this gives the pricing kernel ξ_2. The empirical distribution in case of a peso problem for terminal states is given by $\widetilde{P}^{term}$, and the empirical distribution in case of a path peso problem is given by $\widetilde{P}^{path}$.

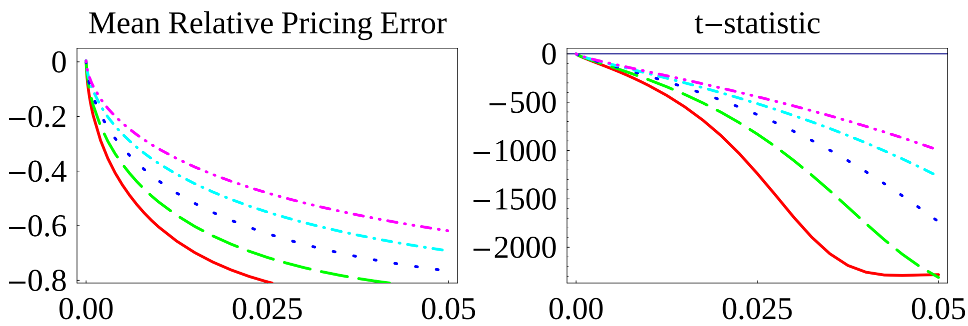

Figure 2: Peso Problem for Terminal States: Classical Martingale Restriction

The figure shows the mean relative pricing error for put options (left graph) and the corresponding t-statistic (right graph) that result from a test of the classical martingale restriction as a function of the probability $P(\Omega \setminus A)$ of missing paths. We assume that the paths with the highest absolute excess return over the interval $[t, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

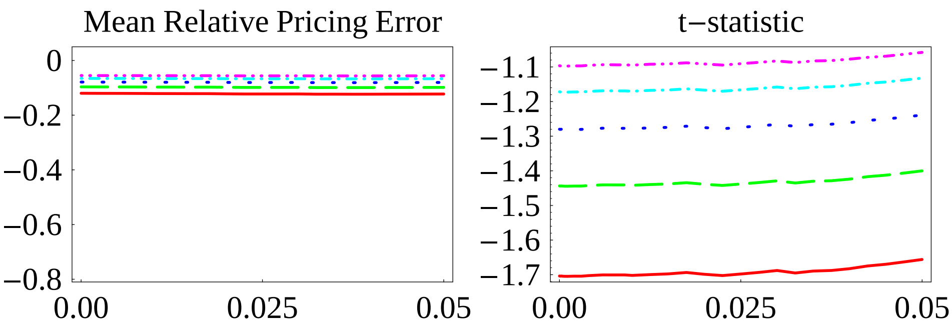

Figure 3: Peso Problem for Terminal States: Restriction of Bondarenko (2003b)

The figure shows the mean relative pricing error for put options (left graph) and the corresponding t-statistic (right graph) that result from a test of the restriction of Bondarenko (2003b) as a function of the probability $P(\Omega \setminus A)$ of missing paths. We assume that the paths with the highest absolute excess return over the interval $[t, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

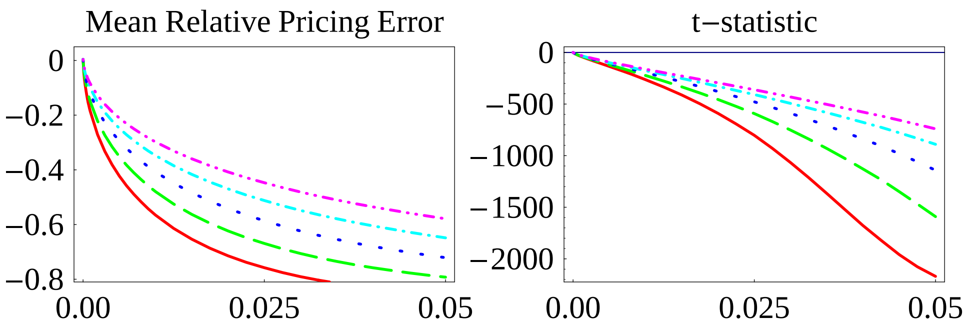

Figure 4: Path Peso Problem: Classical Martingale Restriction

The figure shows the mean relative pricing error for put options (left graph) and the corresponding t-statistic (right graph) that result from a test of the classical martingale restriction as a function of the probability $P(\Omega \setminus A)$ of missing paths. We assume that the paths with the highest variation of the stock price over the periods $[t, u]$ and $[u, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

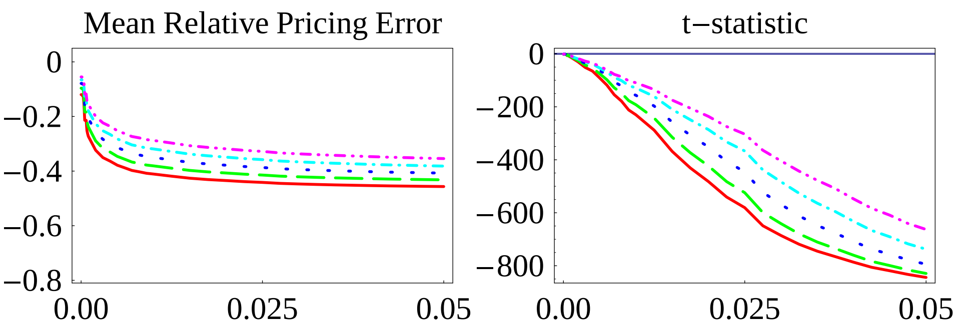

Figure 5: Path Peso Problem: Restriction of Bondarenko (2003b)

The figure shows the mean relative pricing error for put options (left graph) and the corresponding t-statistic (right graph) that result from a test of the restriction of Bondarenko (2003b) as a function of the probability $P(\Omega \setminus A)$ of missing paths. We assume that the paths with the highest variation of the stock price over the periods $[t, u]$ and $[u, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

Figure 6: No systematic pattern of omitted paths

The figure shows the mean relative pricing errors for put options (left column) and the corresponding t-statistics (right column) that result from a test of the classical martingale restriction (upper row) and the restriction of Bondarenko (2003b) (lower row) as a function of the probability $P(\Omega \setminus A)$ of missing paths. The omitted paths are selected randomly. The moneyness (defined as the ratio of strike price over the stock price) of the options is 0.80, 0.84, 0.88, 0.92, and 0.96, and the pricing errors are approximately the same.

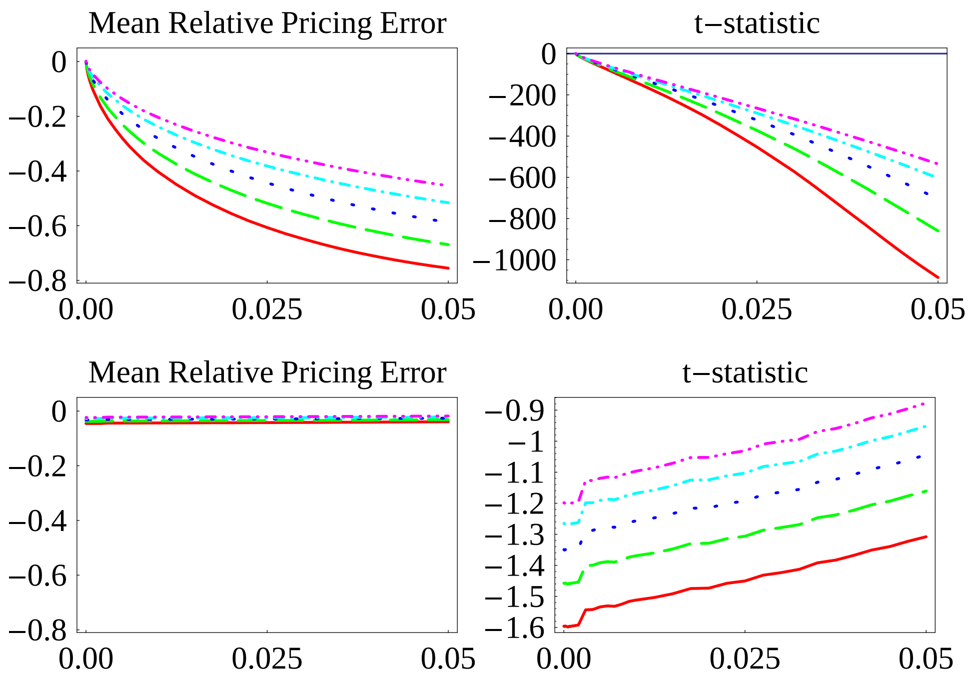

Figure 7: Peso Problem for Terminal States in the Model of Black and Scholes (1973)

The figure shows the mean relative pricing errors for put options (left column) and the corresponding t-statistics (right column) that result from a test of the classical martingale restriction (upper row) and the restriction of Bondarenko (2003b) (lower row) as a function of the probability $P(\Omega \setminus A)$ of missing paths. The true data-generating process is Black and Scholes (1973) with $\sigma = 0.3$. We assume that the paths with the highest absolute excess return over the interval $[t, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

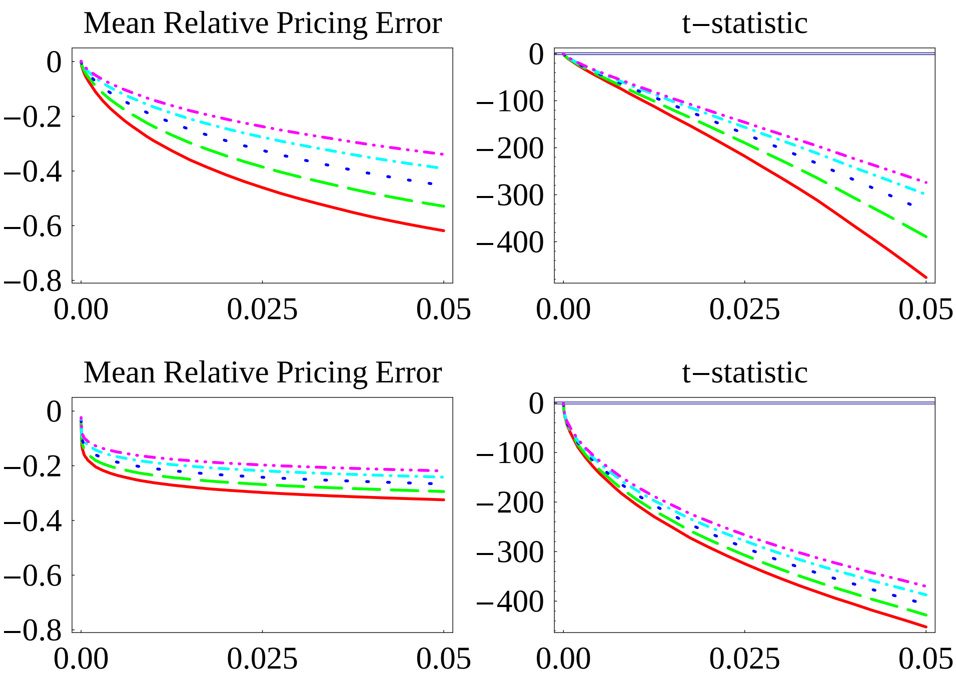

Figure 8: Path Peso Problem in the Model of Black and Scholes (1973)

The figure shows the mean relative pricing errors for put options (left column) and the corresponding t-statistics (right column) that result from a test of the classical martingale restriction (upper row) and the restriction of Bondarenko (2003b) (lower row) as a function of the probability $P(\Omega \setminus A)$ of missing paths. The true data-generating process is Black and Scholes (1973) with $\sigma = 0.3$. We assume that the paths with the highest variation of the stock price over the periods $[t, u]$ and $[u, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

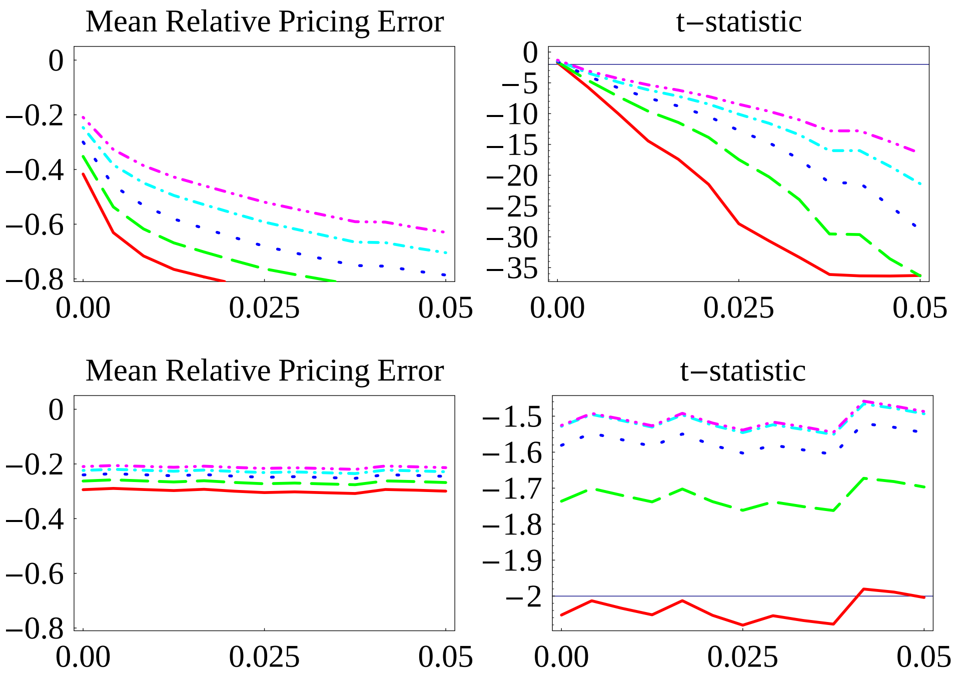

Figure 9: Peso Problem for Terminal States in the Model of Merton (1976) for a Sample Size of 240

The figure shows the mean relative pricing errors for put options (left column) and the corresponding t-statistics (right column) that result from a test of the classical martingale restriction (upper row) and the restriction of Bondarenko (2003b) (lower row) as a function of the probability $P(\Omega \setminus A)$ of missing paths. The true data-generating process is Merton (1976) with $\sigma = 0.3$, the sample size is 240. We assume that the paths with the highest absolute excess return over the interval $[t, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.

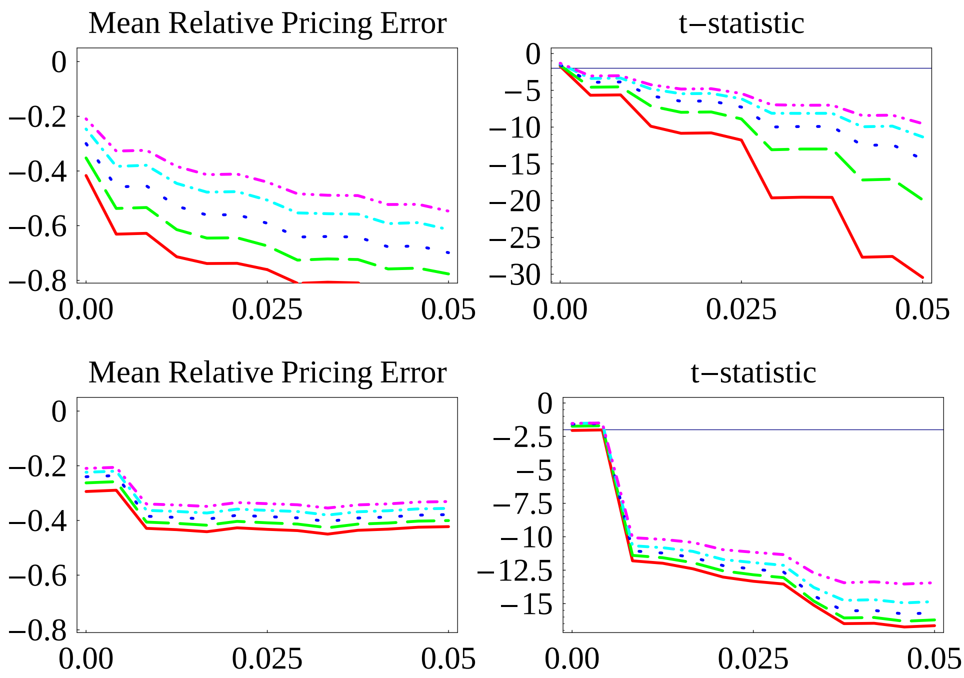

Figure 10: Path Peso Problem in the Model of Merton (1976) for a Sample Size of 240

The figure shows the mean relative pricing errors for put options (left column) and the corresponding t-statistics (right column) that result from a test of the classical martingale restriction (upper row) and the restriction of Bondarenko (2003b) (lower row) as a function of the probability $P(\Omega \setminus A)$ of missing paths. The true data-generating process is Merton (1976) with $\sigma = 0.3$, the sample size is 240 We assume that the paths with the highest variation of the stock price over the periods $[t, u]$ and $[u, T]$ are missing in the sample. The moneyness (defined as the ratio of strike price over the stock price) of the options is, from bottom to top, 0.80, 0.84, 0.88, 0.92, and 0.96, so that the percentage pricing error is the larger in absolute terms the more the option is out of the money.