An Economic Motivation for Variance Contracts

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Our main argument is that the variance contract is superior to a dynamic replication strategy due to parameter risk, and model risk. To show this we analyze the local hedging errors for the variance contract under different scenarios, namely under pure estimation risk (or parameter risk) in a stochastic volatility and in a jump-diffusion model, under model risk when the wrong type of risk factor is assumed to be present (stochastic volatility instead of jumps or vice versa), and under model risk when risk factors are omitted (e.g. when the true model contains jumps which are not present in the model assumed by the investor). The results show that the variance contract is exposed to model risk to an economically significant degree, and that it is much harder to hedge than, e.g., deep OTM puts. We also show that the semi-static replication strategy which builds on the log-contract fails in case of jumps. Our conclusion is that the improvement provided by the introduction of a variance contract is greater than the one offered by the introduction of additional standard options.

Keywords: Variance Risk, Stochastic Volatility, Jump-Diffusion, Model Risk, Parameter Risk, Hedging Error

JEL: G12, G13

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1 Introduction and Motivation

There is ample evidence from empirical studies of option markets that there are additional priced factors beyond plain stock price risk (see, e.g., Bakshi, Cao, and Chen (1997), Bates (2000), Pan (2002), and Eraker (2004)). Therefore, extensions of the seminal model developed by Black and Scholes (1973) contain factors like stochastic volatility (SV) or stochastic jumps (SJ). The presence of these additional factors makes markets incomplete, so that trading in the stock and the money market account is not enough to replicate an arbitrary derivative payoff or to realize the optimal consumption stream. For this purpose more contracts are needed. This insight is probably the most important motivation for the introduction of innovative derivatives like variance contracts, which recently took place on major exchanges like the CBOE, as described in Bondarenko (2004).

A variance contract permits the trading of the stochastic variation of stock return volatility by basically paying off the sum of squared price changes. The larger the realized variance of the stock due to increases in stochastic volatility or jumps in the stock price, the higher the payoff of the variance contract. It therefore allows speculation on the amount of risk represented by the stock over a certain period of time. In this paper, we first provide a detailed discussion of the risk exposure of the variance contract and we discuss different competing explanations for the significantly negative risk premium which is documented in empirical studies, e.g. in Bondarenko (2004). Second, we provide economic arguments for the introduction of the variance contract by an exchange.

The first question we address is why investors would want to trade volatility risk and jump risk in addition to trading stock price risk. The standard answer lists two main motives: speculation and hedging. If the market price of risk for a risk factor is different from zero, the investor trades this risk factor to earn the risk premium. Furthermore, risk factors like stochastic volatility may have an impact on future investment opportunities, that is on future interest rates and/or market prices of risk which then change randomly over time. The investor takes this stochastic behavior into account when planning his portfolio by taking some long or short positions in the respective risk factors, which Merton (1971) called 'hedging demand'. Note, however, that this hedging motive is not exactly the same as the one related to the risk management of a derivative position. Especially, we do not claim that investors mainly use the variance contract to hedge vega risk in their option books.

Liu and Pan (2003) solve a portfolio planning problem in a partial equilibrium model with stochastic volatility and jumps (of deterministic size) in the stock price. Branger, Schlag, and Schneider (2005b) also include simultaneous jumps in volatility. They consider a complete market and derive the investor's optimal exposure to stock price risk, volatility risk, and jump risk. Their results show that the investor wants to trade both stochastic volatility and jump risk, and their results also allow to see under what circumstances different investors would want to hold long and short positions in volatility risk. Branger, Schlag, and Schneider (2005a) analyze a general equilibrium setup and show that heterogeneous agents will trade both stochastic volatility and jump risk. These findings confirm that there is indeed a demand for positions in variance risk.

Given this trading demand of the investors, the main part of our paper is dedicated to the question why investors might be interested in the introduction of additional contracts, and why they might prefer the introduction of the variance contract to an augmentation of the set of available options. A natural first step seems to be to distinguish between a complete and an incomplete market. If the market is incomplete, one reason for the introduction of a variance contract could be that it is market-completing. However, this would also be true for an additional option. When too few contracts are traded, why should one introduce a rather complicated derivative asset like a variance contract instead of additional simple European options? On the other hand, if the market is complete, the variance contract can be replicated by a dynamic trading strategy, and this also holds for other new claims. There is no need to introduce further redundant claims, which in addition would only take away liquidity from existing contracts. Summing up: In an incomplete market the variance contract does not seem to be the most natural contract to introduce, and in a complete market, there is no need to introduce it.

We thus have to explain why the variance contract is not necessarily redundant, i.e. why perfect replication is not possible, and we have to explain why there is more need to introduce the variance contract than some standard option. In this paper, we argue that the 'replicating' strategy may fail due to model risk, which makes it impossible for the investor to know the correct composition of the hedge portfolio. In these cases, the variance contract will be better than its 'replicating' strategy, because the investor does not have to worry about the potentially difficult replication, but can buy the desired payoff directly.

Our ultimate line of argument for the variance contract then consists of three steps. First, a new contract should only be introduced if investors want to trade the payoff profile or risk exposure provided by this contingent claim. We argue that this condition holds for the variance contract. Second, the introduction makes sense only if the 'replicating' strategy for the new contract is subject to model risk to an economically significant extent, and we show that this is the case for the variance contract. Third, if there is still more than one contract that could be introduced, the choice should be the one which is hardest to replicate. The main result of our analysis is indeed that the variance contract is much harder to replicate than standard derivatives like a deep OTM put option.

We argue that the 'replicating' strategy for the variance contract may fail due to model risk. The true data generating process is not known so that the replicating strategy will be determined in some model (the hedge model), which is in general not equal to the true model. Thus, the dynamic replication strategy will be more risky than trading the contract itself. Model risk can occur in different forms and to different degrees, and we consider both the impact of parameter risk and model mis-specification. Parameter risk means that the correct type of model is used, but with incorrect parameters (possibly derived from some estimation, so that parameter risk is basically equivalent to estimation risk). A second form of model risk occurs when the wrong risk factors are included in the hedge model. The most prominent example is to use a hedge model which contains a stochastic volatility component, while the true model exhibits stochastic jumps (or vice versa). Finally, risk factors present in the true model may be omitted in the hedge model, so that instead of stochastic volatility and stochastic jumps together only one of these risk factors is included in the hedge model.

Besides dynamic hedging strategies, we have to consider a second class of hedging strategies for the variance contract, namely semi-static replication strategies discussed, e.g., by Neuberger (1994), Bondarenko (2004), Carr and Wu (2004) and Jiang and Tiang (2005). They show that for an underlying without jumps the variance contract is equal to a position in log-contracts (which can be statically replicated via a continuum of options), the money market account and some dynamic strategy in the underlying. In practice, this approach suffers from at least two problems. The first is the assumption that a continuum of options is available. Even if enough options are traded to *price* the variance contract, this does not necessarily imply that the static *hedge* of the log-contract is of similar accuracy. Second, the hedge is perfect only for a certain class of models with deterministic interest rates and without jumps in the stock price, or, as shown by Bondarenko (2004) for the case of jumps, if the contract specification is adjusted appropriately. Jump risk is thus a problem and can only be dealt with by dynamic hedging strategies which are exposed to model risk.

Finally, it seems useful to consider the perspective of 'normal' investors on the one hand and highly specialized financial institutions like hedge funds on the other hand. Normal investors interested in trading variance risk will prefer the direct investment into a variance contract to a dynamic 'replicating' strategy which may fail due to model risk, and also to the semi-static strategy described above, which is still exposed to jump risk. Highly specialized investors will have an advantage in replicating the variance contract and will therefore provide liquidity in this market. They use trading strategies which involve a wide array of instruments anyway and which are based on highly sophisticated models. It thus seems reasonable to assume that they will be quite successful in hedging the variance contract. Nevertheless, they are still exposed to model risk (to a smaller extent than 'normal' investors), and they will basically compete with each other in terms of their modeling and hedging competence.

To analyze the impact of model risk on the replicating strategy we derive analytical expressions for local hedging errors over an infinitesimal time interval. The global hedging error over some interval from t to $t+\tau$ is then simply given by the integral over these local errors, so that an analysis of local errors seems sufficient. We show that the hedging error depends on the error in the sensitivities of the claim to be hedged with respect to the risk factors, as well as on the error in the sensitivities of the hedge instruments. We also derive a robustness condition, under which the strategy would have a zero exposure to model risk, and which basically says that the two errors mentioned above should exactly offset each other. Intuitively, we expect model risk to be only a moderate problem when the claim to be hedged and the hedge instrument are very similar. To achieve a robust hedge for the variance contract, there are two natural candidates, namely an ATM straddle (or ATM option) and an OTM put. The ATM straddle is often considered an ideal instrument for volatility trading and could thus be used to hedge the exposure of the variance contract with respect to volatility risk. The OTM put allows for the trading of downside jump risk and could therefore be used to hedge the jump risk exposure of the variance contract.

To concentrate on the main issues we make several simplifying assumptions. First, we restrict our analysis to the case where only the stock, the money market account, and one further option are already traded, and we choose the hedge models such that the market is complete with this set of basis assets. The investor then uses the (unique) replicating strategy from the hedge model. In an incomplete market, one would have to decide which hedge criterion (like super-hedging, variance minimization or quantile hedging) to use. In a complete market where too many assets are traded, on the other hand, one would have to decide which assets to use in a dynamic strategy or which of the many replicating strategies to use. In both cases, the performance of the hedging strategies would then depend on these additional choices. By our assumptions we avoid this dependence, which allows us to focus directly on the impact of model mis-specification.

We rely on numerical examples to compare the exposures of different claims to model risk. In these examples the true model is given by Merton (1976) including jumps, the stochastic volatility model of Heston (1993), or the model of Bakshi, Cao, and Chen (1997) which includes both stochastic volatility and jumps. We assume that the jump sizes are deterministic. The hedge model (which is either Merton (1976) or Heston (1993)) is then calibrated to the prices of selected options, where we allow for small price deviations well within the bid-ask-spread.

We will now give a brief summary of our results. In our hedging experiments the stock, the money market account, and standard European options with varying strike prices are used as hedge instruments. We compare the hedging error for the variance contract to the hedging error for a deep out-of-the-money (OTM) put option. We have chosen this instrument as the benchmark case, since it represents an example for a standard claim, which might compete with the variance contract for introduction on a derivatives exchange. It is not surprising that the hedging error for this OTM put is the smaller the lower the difference between its strike and the strike of the option we use as the hedge instrument, which means that we can actually choose some best hedge instrument among the available contracts. For the variance contract the hedging errors are comparable in size to those observed for the benchmark put. However, the variance contract is more difficult to hedge for two reasons. First, there is no ideal hedge instrument for which the hedging error due to model risk would vanish. Second, a hedge instrument which provides a good hedge for the variance contract in one scenario may perform rather badly in other scenarios, so there is no dominant choice of hedge instruments. In fact, it turns out that neither the ATM straddle nor the OTM put provide a robust hedge. Thus, the variance contract is exposed to model risk much more than a deep OTM put option. The actual introduction of the variance contract is an improvement over the situation when investors have to replicate its payoff using a traded option only, and it offers a more significant improvement than a deep OTM put. In a nutshell, the variance contract provides the easiest way to generate a positive exposure to increasing realized stock variation irrespective of the true model and without sensitivity with respect to the stock price.

The remainder of the paper is structured as follows. In Section 2 we analyze the variance contract with respect to its pricing and the risk factors its holder is exposed to.

In Section 3 we discuss why investors want to trade variance risk. Section 4 discusses the performance of semi-static and dynamic hedging strategies under model mis-specification. Section 5 concludes.

2 Risk Factors Affecting Variance Contracts

The first step in analyzing a new contract is to investigate its exposure to the different risk factors in the given model. We show that, possibly in contrast to intuition, the 'variance risk' captured by the variance contract is not just stochastic volatility, and that the risk premium earned is not just a premium for stochastic volatility.

2.1 Model Setup

We use a model with stochastic volatility and jumps in both the stock price and in volatility. This type of model has been investigated by Duffie, Pan, and Singleton (2000), Eraker (2004), and Broadie, Chernov, and Johannes (2005). The stochastic processes for the state variables under the true measure P are given by the stochastic differential equations

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sqrt{V_t} S_t dW_t^{(S)} + S_{t-} \left\{ \left(e^{X_t} - 1\right) dN_t - E^P \left[e^X - 1\right] k_t^P \, dt \right\} \qquad (1) \\ dV_t &= \kappa^P \left(\theta^P - V_t\right) dt + \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) \\ &\quad + \left\{ Y_t dN_t - E^P \left[Y\right] k_t^P \, dt \right\}. \qquad (2) \end{aligned}$$

The interest rate is constant and denoted by r. The drift μ_t of the stock depends on the market prices of risk as explained below. The intensity of the jump process N under P is given by $k_t^P = k_0^P + k_1^P V_t$ where we assume $k_0^P \geq 0$ and $k_1^P \geq 0$ to avoid negative jump intensities. X denotes the jump size in the log of the stock price, and Y is the jump in variance, where we assume $Y \geq 0$ to avoid a negative variance. We make the simplifying assumptions that jumps in volatility occur simultaneously with jumps in the stock price, and that the jump sizes are uncorrelated.

With the stock and the money market account only, the market is incomplete. The market prices of risk are not unique, but have to be given exogenously. We assume that the compensation per unit of $\sqrt{V_t} dW_t^{(S)}$ is given by $\lambda^{(S)} V_t$, while $\lambda^{(V)} V_t$ is the average reward for bearing one unit of $\sqrt{V_t}\left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)}\right)$. The jump intensity changes to $k_t^Q = k_0^Q + k_1^Q V_t$ under the risk-neutral measure Q, where $k_0^Q \geq 0$, $k_1^Q \geq 0$. The jump size in the stock has the new mean $E^Q[e^X - 1] = \mu_X^Q$, and the mean jump size in volatility is given by $E^Q[Y] = \mu_Y^Q$.

Under the risk-neutral measure Q the processes are

$$\begin{aligned} dS_t &= r S_t dt + \sqrt{V_t} S_t d\widetilde{W}_t^{(S)} + S_{t-} \left\{ \left(e^{X_t} - 1\right) dN_t - E^Q \left[e^X - 1\right] k_t^Q \, dt \right\} \\ dV_t &= \kappa^Q \left(\theta^Q - V_t \right) dt + \sigma_V \sqrt{V_t} \left(\rho d\widetilde{W}_t^{(S)} + \sqrt{1-\rho^2} d\widetilde{W}_t^{(V)} \right) + \left\{ Y_t dN_t - E^Q \left[Y\right] k_t^Q \, dt \right\}, \end{aligned}$$

where the mean reversion of volatility and the mean level of variance under Q are given by

$$\begin{aligned}\kappa^Q &= \kappa^P + \lambda^{(V)}\sigma_V + E^P[Y]k_1^P - E^Q[Y]k_1^Q \\ \kappa^Q\theta^Q &= \kappa^P\theta^P - E^P[Y]k_0^P + E^Q[Y]k_0^Q.\end{aligned}$$

The local expected return μ_t on the stock is given by

$$\mu_t \equiv r + \lambda^{(S)}V_t + E^P\left[e^X - 1\right]k_t^P - E^Q\left[e^X - 1\right]k_t^Q.$$

The second summand on the right hand side gives the compensation for diffusion risk, while the last two terms are the compensation for jump risk. For a contingent claim exposed to dV, like a call option, there would also a compensation for the diffusion risk in V, given by $\sigma_V\lambda^{(V)}V_t$ times the amount of risk, and a premium for jump risk in volatility, which again depends on the difference in the intensity of the jump and the jump size distribution between P and Q.

2.2 Pricing the Variance Contract

The payoff C_T of a derivative on variance at its maturity date T depends on the realized variance $RV(0,T)$ of the stock over the time interval $[0,T]$ and is thus highly path-dependent. Derivatives on variance include, e.g., variance swaps, volatility swaps or options on variance and volatility. In the following, we will focus on a generic variance contract with payoff given by $RV(0,T)$.

When discrete returns are used the realized variance is

$$RV(0,T) = \int_0^T \left(\frac{dS_t}{S_{t-}}\right)^2 = \int_0^T V_u du + \int_0^T \left(e^{X_u} - 1\right)^2 dN_u,$$

while in the case of log-returns the payoff is given by

$$RV(0,T) = \int_0^T (d\ln S_t)^2 = \int_0^T V_u du + \int_0^T X_u^2 dN_u. \tag{3}$$

The first integral is the accumulated variance of the diffusion component in stock returns (the extension to multiple diffusions is straightforward), i.e. the larger V, the larger the payoff. The second integral represents the sum of squared jumps in the stock price (note that the sign of the jumps does not matter). If there are sudden large changes in the stock price, the payoff of the variance contract also increases. In what follows we assume continuous monitoring and do not discuss problems related to discretization error or measurement error, as it is done, e.g., in Bondarenko (2004) and Carr and Wu (2004). Furthermore, we assume that $RV(0,t)$ is observable, and we focus on the case of log returns, i.e. on Equation (3).

The price at time t of the variance contract for continuously monitored log returns is given by

$$\begin{aligned} C_t &= E^Q\left[e^{-r(T-t)}\left(\int_0^T V_u du + \int_0^T X_u^2 dN_u\right) \Big| \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\Bigg\{RV(0,t) + E^Q[X^2]k_0^Q(T-t) \\ &\quad + \left(1 + k_1^Q E^Q[X^2]\right)\left[\theta^Q\ (T-t) + \frac{1-e^{-\kappa^Q(T-t)}}{\kappa^Q}\left(V_t - \theta^Q\right)\right]\Bigg\}, \end{aligned} \tag{4}$$

A proof of the pricing formula is given in Appendix A.1. For discrete returns, one can simply replace X by $e^X - 1$ in Equation (4).

At time t, the variance contract over the period $[0,T]$ can be decomposed into an investment in the money market account and an investment in a variance contract over $[t,T]$. The price of the future realized variance, $RV(t,T)$, depends on the distribution of jumps in the stock price (irrespective of their signs), on their intensity, and on the expected volatility over the life of the contract. The mixed term, where both jumps and volatility show up, is due to a volatility dependent jump intensity and vanishes for $k_1^Q = 0$.

2.3 Risk Premia for the Variance Contract

The variance contract is delta-neutral by construction, so its sensitivity with respect to the stock price is zero. The partial derivative of the claim price $C_t = c(t, V_t, RV(0,t), \dots)$ with respect to volatility is given by

$$\frac{\partial c}{\partial v} = e^{-r(T-t)}\left(1 + k_1^Q E^Q[X^2]\right)\frac{1-e^{-\kappa^Q(T-t)}}{\kappa^Q} > 0,$$

so that the exposure to volatility risk is always positive. Jumps in the stock price and in volatility also have an impact on C. The price change due to a simultaneous jump in the stock price and volatility is given by

$$\Delta C_t = e^{-r(T-t)}X_t^2 + e^{-r(T-t)}\left(1 + k_1^Q E^Q[X^2]\right)\frac{1-e^{-\kappa^Q(T-t)}}{\kappa^Q}Y_t \geq 0.$$

The first term on the right hand side represents the impact of a jump in the stock price, increasing the accumulated payoff $RV(0,t)$. The second term captures the impact of a jump in volatility, increasing the price of future realized variance $RV(t,T)$.

The exposures of the variance contract can, of course, also be generated via some dynamic trading strategy in the stock, the money market account, and a sufficient number of contingent claims. An easy example is provided by a straddle where the strike is chosen such that it is delta-neutral. However, the strike price depends on the model and on its parameters, and the portfolio has to be adjusted continuously. The variance contract, on the other hand, is delta-neutral by construction, and this property holds irrespective of the model.

The difference between the drifts of C under P and Q is the local risk premium on the variance contract:

$$\begin{aligned}
& E^P[dC_t|\mathcal{F}_t] - E^Q[dC_t|\mathcal{F}_t] \\
= \quad & e^{-r(T-t)}\Big\{ \Big(1 + k_1^Q E^Q[X^2]\Big) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \lambda^{(V)} \sigma_V V_t \\
& + \Big(E^P[X^2] k_t^P - E^Q[X^2] k_t^Q \Big) \\
& + \Big(1 + k_1^Q E^Q[X^2]\Big) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \Big(E^P[Y] k_t^P - E^Q[Y] k_t^Q \Big) \Big\} dt.
\end{aligned}$$

A proof is given in Appendix A.2. The premium can be decomposed into a compensation for volatility diffusion risk (depending on $\lambda^{(V)}$), for jump risk in the stock price (depending on the expectation of squared jumps and on the jump intensity), and for volatility jump risk (depending on the expected jump size and on the jump intensity). Note that there is a premium for quadratic, but not for linear stock price risk, a consequence of the fact that the variance contract is delta-neutral. Furthermore, note that the expected excess return does not depend on the *sign* of the jumps, but only on their absolute size. The reason is, of course, that jumps increase the volatility of the underlying, no matter whether they are upward or downward.

This decomposition of the risk premium shows that there are several alternative explanations for the empirically observed negative premium on the variance contract (as found in Bondarenko (2004)). First, the negative risk premium can be attributed to volatility diffusion risk. Given $\sigma_V > 0$ and $V > 0$, as in Heston (1993), this implies that the market price of volatility risk $\lambda^{(V)}$ is negative. Second, the negative risk premium can be explained by a premium for jump risk in the stock price. Under the realistic assumption that jumps are on average negative, we expect the jump intensity to be larger under the risk-neutral measure than under the true measure. If, in addition, jumps are also more severe under Q than under P, then the contribution of stock jump risk to the risk premium on the variance contract is negative, too. Finally, the risk premium can be attributed to volatility jumps. In this case, a negative risk premium arises if jumps are more frequent and more severe under the risk-neutral measure than under the true measure.

For empirical studies, it is important to keep in mind these alternative explanations for a negative risk premium on the variance contract. As shown above, the variance contract does not allow to distinguish between a volatility and a jump risk premium.

3 Motives for Trading Variance Contracts

If a contract is to be newly introduced on an exchange or as an OTC derivative, an important condition for its success is that it provides a payoff profile or risk exposure that investors actually want to trade. For the variance contract the question is thus whether the investor is interested in an exposure to the second moment of stock returns.

As the theoretical basis for our analyses we rely on the work of Liu and Pan (2003) and the extension by Branger, Schlag, and Schneider (2005b). The latter paper analyzes the portfolio planning problem of an investor with a CRRA utility function in a model which is characterized by SV and jumps in both the stock price and in volatility. To simplify the analysis, we assume that jumps occur simultaneously and that their size is deterministic. The dynamics of the stock price and the SV component are given in Equations (1) and (2), again with the additional assumptions that the jump sizes are deterministic, i.e. $e^X - 1 \equiv \mu_X$ and $Y \equiv \mu_Y$. Furthermore, we assume that the jump intensities are proportional to V, i.e. $k_0^P = k_0^Q = 0$. The risk premium for one unit of $dW_t^{(S)}$ is given by $\eta^{B1}\sqrt{V_t}$, the compensation for $dW_t^{(V)}$ is $\eta^{B2}\sqrt{V_t}$. The market prices of risk introduced in Section 2.1 are then given by

$$\begin{aligned} \lambda^{(S)} &= \eta^{B1} \\ \lambda^{(V)} &= \rho\eta^{B1} + \sqrt{1-\rho^2}\eta^{B2}. \end{aligned}$$

In the above setup the market is complete with two additional instruments besides the stock and the money market account. Branger, Schlag, and Schneider (2005b) solve the portfolio planning problem for an investor with constant relative risk aversion (power utility). The dynamics of wealth W are

$$\begin{aligned} dW_t &= W_t \Big\{ r dt + \theta_t^{(S)} \left(\sqrt{V_t} dW_t^{(S)} + \eta V_t dt \right) + \theta_t^{(V)} \left(\sqrt{V_t} dW_t^{(V)} + \xi V_t dt \right) \\ &\quad + \theta_t^{(N)} \left[dN_t - k_1^P V_t \, dt + \left(k_1^P - k_1^Q \right) V_t \, dt \right] \Big\} \end{aligned}$$

where the exposures to the three fundamental sources of risk $\sqrt{V_t}W^{(S)}$, $\sqrt{V_t}W^{(V)}$, and N are used instead of the investments in the different assets. In a complete market, there exists a dynamic trading strategy for every possible exposure.

As part of the solution the optimal exposure is determined as

$$\begin{aligned} \theta_t^{*S} &= \frac{\eta^{B1}}{\gamma} + \rho\sigma_V H(\tau) \\ \theta_t^{*V} &= \frac{\eta^{B2}}{\gamma} + \sqrt{1-\rho^2}\sigma_V H(\tau) \\ \theta_t^{*N} &= \left[\left(\frac{k_1^P}{k_1^Q} \right)^{1/\gamma} - 1 \right] + \left(\frac{k_1^P}{k_1^Q} \right)^{1/\gamma} \left[e^{H(\tau)\mu_Y} - 1 \right], \quad 1 + \theta_t^{*N} \geq 0. \end{aligned}$$

with γ as the investor's coefficient of risk aversion and τ as the remaining investment horizon. The function H solves the ordinary differential equation

$$H'(\tau) = a + b \exp\{\mu_Y H(\tau)\} + c\, H(\tau) + d\, H^2(\tau) \tag{5}$$

with the boundary condition $H(0) = 0$ and

$$\begin{aligned}
a &= \frac{1-\gamma}{2\gamma^2}\left[(\eta^{B1})^2 + (\eta^{B2})^2\right] + \frac{1-\gamma}{\gamma}\, k_1^Q - \frac{1}{\gamma}\, k_1^P \\
b &= k_1^Q \left(\frac{k_1^P}{k_1^Q}\right)^{1/\gamma} \\
c &= -\left(\kappa^P + \mu_Y k_1^P\right) + \frac{1-\gamma}{\gamma}\sigma_V \left(\rho\eta^{B1} + \sqrt{1-\rho^2}\eta^{B2}\right) \\
d &= \frac{1}{2}\,\sigma_V^2.
\end{aligned}$$

The optimal exposure can be decomposed into a speculative demand (first summand) and a hedging demand (second summand). For the diffusion components the speculative demand depends on the ratio of the risk premium to the coefficient of risk aversion, while the hedging demand is mainly driven by the correlation between stock returns and volatility changes. For the jump component, there is also a speculative demand, and the hedging demand vanishes only in case $\mu_Y = 0$, that is when jumps have no impact on the state variable V. For a more detailed discussion of the optimal demand, the reader is referred to Liu and Pan (2003) and Branger, Schlag, and Schneider (2005b).

If only the stock and the money market account are traded, the investor cannot achieve an exposure to $W^{(V)}$, and the exposures to $W^{(S)}$ and N are fixed at the relation given by the stock price process. To obtain his optimal exposure, the investor thus has to use derivatives, which shows that there is a need for contracts providing exposure to the individual risk factors. Depending on the level of risk aversion and the amount of the risk premia, the sign of the optimal exposure can vary, i.e. both long and short positions in the risk factors can basically be optimal. While this analysis is performed in a partial equilibrium model, Branger, Schlag, and Schneider (2005a) show that there is also a trading demand for variance risk in a general equilibrium setup with heterogeneous agents.

4 Replication Strategies

As mentioned in the introduction, a new derivative will only be introduced if it cannot be easily replicated (at least not by a sufficiently large group of investors) by claims that are already traded in the market. In this paper, we focus on the degree of sensitivity to model risk as a key parameter in measuring the economic value of a newly introduced derivative contract. Even if the contract was basically replicable in a scenario with perfect knowledge about the true data-generating process, the hedge errors occurring due to a mis-specified hedge model may nevertheless be substantial. In this case investors might prefer trading the variance contract itself to the supposedly replicating strategy which turns out to be more risky. However, this argument not only holds for the variance contract, but also for other derivatives. Which contract should ultimately be introduced depends on the amount of model risk exposure.

Before we analyze the impact of model risk on dynamic replicating strategies, we take a closer look at semi-static hedging strategies for the variance contract. If the stock price follows a diffusion process, interest rates are deterministic, and if a continuum of options with strike prices from zero to infinity are available for trading, then there is a perfect and model-independent hedge for the variance contract. This hedge, however, fails if the stock price can jump.

4.1 Semi-Static Replication

Since the paper of Breeden and Litzenberger (1978), it is well known that a European path-independent claim can be replicated by a static portfolio of a continuum of European call options with the same time to maturity. Furthermore, its decomposition is model-independent and consequently not exposed to model risk.

The principle of static hedging cannot be applied directly to variance contracts, since their payoff is highly path-dependent. In a diffusion model, however, it can be shown that the payoff of the variance contract is equal to the payoff from a log-contract (for which a static replication is possible) and a simple dynamic strategy in the stock, c.f. Neuberger (1994). This hedge is perfect for diffusion models, but it fails if the stock price can jump or if interest rates are stochastic. In the following, we will concentrate on the impact of jumps, but retain the assumption of deterministic interest rates.

In our general model introduced in Section 2.1, it holds that

$$d\ln S_t - \frac{dS_t}{S_{t-}} = -0.5V_t dt + \left[X_t - \left(e^{X-t} - 1\right)\right] dN_t.$$

We can thus conclude that

$$\begin{aligned}\int_0^T V_t dt + \int_0^T X_t^2 dN_t &= -2(\ln S_T - \ln S_0) + 2\int_0^T \frac{dS_t}{S_{t-}} \\ &\quad + 2\int_0^T \left[X_t + 0.5X_t^2 - \left(e^{X-t} - 1\right)\right] dN_t,\end{aligned}$$

which is linked to the 'static replicating strategy' for the variance contract in case of log-returns. For discrete returns, we get

$$\begin{aligned}\int_0^T V_t dt + \int_0^T \left(e^{X_t} - 1\right)^2 dN_t &= -2(\ln S_T - \ln S_0) + 2\int_0^T \frac{dS_t}{S_{t-}} \\ &\quad + 2\int_0^T \left[X_t + 0.5\left(e^{X_t} - 1\right)^2 - \left(e^{X-t} - 1\right)\right] dN_t\end{aligned}$$

The first term on the right hand side is in both cases related to the log-contract. It can be replicated by a static position in a continuum of European options with maturity in T and by a position in the money market account. The second term on the right hand side can be replicated by a dynamic trading strategy in the stock or in the futures contract, where the number of stocks (futures) depends on the current stock (futures) price and on the interest rate.

If there are no jumps, the third term is zero in both cases, and the strategy just described is a perfect hedge for the variance contract. In case of a jump, however, this term deviates from zero and causes the replication strategy to fail. Figure 1 shows this replication error as a function of the jump size $e^{X_t} - 1$ in the stock price. The left panel describes the error if the realized variance is based on log returns, and the right panel gives the error if discrete returns are used. The upper row shows that the error is quite low for jumps below 10%, but increases sharply for more extreme jumps. To get an idea of the severity of the problem, we compare the hedging error due to jumps to several benchmarks. First, the final payoff due to diffusion variance is on average equal to $\theta^P \tau$ where τ is the time to maturity. For $\theta^P = 0.02$, an error of 0.002 is equal to the payoff due to diffusion variance accumulated over 0.1 years. A jump contributes X_t^2 or $(e^{X_t} - 1)^2$ to the final payoff. As shown in the middle row, the hedging error is equal to approx. 2-8% of this payoff. Second, we can take the price of the contract as a benchmark. For a time to maturity of one month, the error in case of a jump of 20% is equal to up to 5% (in case of log returns) or 20% (in case of discrete returns) of the price of the contract, as can be seen from the bottom row. While the error is thus quite small compared to the total payoff in case of a jump (and thus has a small contribution to the price, as also shown in Jiang and Tiang (2005)), it is quite large compared to the initial price of the contract.

Bondarenko (2004) discusses a modification of the payoff function which is adjusted in such a way that the hedge described above is perfect even in case of jumps (but not in case of stochastic interest rates). For all other specifications of the contract, hedging the jump part of realized variance is still a problem. And in most cases it will be the payoff which is given first (and for which a hedging strategy has to be found) rather than the other way around.

4.2 Basic Setup for Dynamic Hedging Strategies

In the following sections we analyze the performance of dynamic hedging strategies for the variance contract under various types of model risk. We furthermore analyze the hedging errors for a deep OTM put option with a strike price equal to 85% of the current stock price. This option is one example for a contract that competes with the variance contract for introduction. It also completes the market in the sense that it enlarges the set of statically replicable payoffs. From an economic point of view, it provides crash protection and would thus be of interest to investors. Our results show that the variance contract is indeed significantly exposed to model risk, and that it has a higher exposure than the put. Thus, if there is a need for an enlargement of the set of traded contracts, the variance contract offers a more significant improvement than just another put option.

The objective is to replicate some claim H, and in the following this H will either be an OTM-put or the variance contract. S_t is the current stock price, the price of the claim to be hedged is $H_t = h(t, S_t, V_t, \ldots)$, and $C_t^{(i)} = c^{(i)}(t, S_t, V_t, \ldots)$ stands for the price of the i-th hedge instrument written as a function of the state variables. The number of shares in the hedge portfolio at time t is denoted by $\phi_t^{(S)}$, and the number of units of the hedge instrument i at time t is $\phi_t^{(i)}$ for $i = 1, \ldots, n$. If there is only one instrument we denote its price by C and the associated number of units by $\phi_t^{(C)}$. Except for time, partial derivatives are denoted by subscripts.

Besides the true model describing the dynamics of the stock price and the state variables, there is a model assumed by the investor (hedge model). Whenever the calculation of the prices or the portfolio composition is done in this hedge model, we denote the variables by a tilde. For example, $\widetilde{\phi}^{(S)}$ is the number of shares of the stock in the hedge portfolio as calculated in the hedge model.

The hedge portfolio is chosen in such a way that its sensitivities (as calculated in the hedge model) with respect to the different risk factors are equal to those of the claim to be hedged. We make the assumption of market completeness which will be discussed below. For the diffusion risk of the stock this means

$$\widetilde{\phi}^{(S)} 1 + \widetilde{\phi}^{(1)} \widetilde{c}_s^{(1)} + \ldots + \widetilde{\phi}^{(n)} \widetilde{c}_s^{(n)} \quad = \quad \widetilde{h}_s, \tag{6}$$

while for the state variable variance V we must have that

$$\widetilde{\phi}^{(1)} \widetilde{c}_v^{(1)} + \ldots + \widetilde{\phi}^{(n)} \widetilde{c}_v^{(n)} \quad = \quad \widetilde{h}_v. \tag{7}$$

The analogous condition for jump risk is

$$\widetilde{\phi}^{(S)} \Delta \widetilde{S} + \widetilde{\phi}^{(1)} \Delta \widetilde{c}^{(1)} + \ldots + \widetilde{\phi}^{(n)} \Delta \widetilde{c}^{(n)} \quad = \quad \Delta \widetilde{h}, \tag{8}$$

where Δh and $\Delta c^{(i)}$ denote the change in the prices of the claim and the i-th hedge instrument, respectively. By subtracting Equation (6) (multiplied by $\Delta \widetilde{S}$) from Equation (8), this condition can also be rewritten as

$$\widetilde{\phi}^{(1)} \left(\Delta \widetilde{c}^{(1)} - \widetilde{c}_s^{(1)} \Delta \widetilde{S} \right) + \ldots + \widetilde{\phi}^{(n)} \left(\Delta \widetilde{c}^{(n)} - \widetilde{c}_s^{(n)} \Delta \widetilde{S} \right) \quad = \quad \left(\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S} \right). \tag{9}$$

The terms in brackets denote the additional jump risk of the derivatives that remains after delta-hedging with the stock. Of course, we would have a condition like this for any possible jump size (or for any jump size we decide to hedge).

The equations above are given for the hedge model. The same conditions determine the correct replicating portfolio in the true model. However, note that the number of risk factors and the type of risk factors need not be the same in the true and in the hedge model.

As the data-generating process, we consider simple extensions of the Black-Scholes model. To be specific, we work with the jump-diffusion model developed by Merton (1976), the SV model of Heston (1993), and the general model suggested by Bakshi, Cao, and Chen (1997) (assuming a constant interest rate), which are the most prominent models that include stochastic volatility and/or jump risk. The analysis would only become more involved in more complicated models.

As a hedge model, we use the SV model of Heston (1993) and the jump-diffusion model developed by Merton (1976) where we make the simplifying assumption of a deterministic jump size. The hedge model is calibrated to the prices of certain options which are calculated in the true model. For our examples, we rely on European options with moneyness (strike-to-spot ratio) of 0.95, 1, 1.05, and (in case of the SV model) also 0.9, and with a time to maturity of six months which is equal to the time to maturity of the claim to be hedged. We allow for a non-perfect fit of the hedge model to the data, i.e. a hedge model is considered acceptable as long as the maximum deviation of model prices from given market prices is not too large. The investor might consider the hedge model as a simple approximation to the much more complicated true model. Furthermore, real world market frictions like bid-ask spreads could make it almost impossible to infer the exact model and/or the exact parameters anyway, as shown by Dennis and Mayhew (2004). Given the cross section of noisy prices, the investor thus cannot avoid parameter risk and model risk.

The choice of the two hedge models is motivated by several criteria. Our main point is to analyze the impact of model mis-specification on the hedging error. To focus on this question, we make several simplifying assumptions. First, we assume that the hedge model is complete. In an incomplete hedge model, we would have to decide on a hedge criterion like the variance of the hedging error or the shortfall probability. This would introduce an additional dependence of the hedging error on this choice without contributing to the main problem analyzed in the study. Second, we assume that there are just enough hedge instruments to achieve market completeness. Otherwise, the replicating strategy would no longer be unique, and we would have to decide on some additional criterion to choose one of these strategies. Again, this would introduce an additional dependence of the hedging error on this choice and would distract from the main focus of the study. And finally, we assume that one additional option besides the stock and the money market account is enough to complete the market. Thus, all hedging strategies we consider in the following use the same number of instruments, which again facilitates their comparison.

The value of the hedge portfolio at time t is denoted by Π_t. We assume that the hedge portfolio is self-financing, i.e. any proceeds are invested into the money market account, earning the constant risk-free rate r. At time t the hedging error is $D_t = H_t - \Pi_t$. To compare the hedge based on the hedge model to the perfect hedge, we calculate the local hedging error, i.e. we derive the stochastic differential equation (SDE) for the hedging error. As will be shown in more detail below, the general structure of the hedging error is given by

$$\begin{aligned} dD_t \; = \; & \ldots dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} S_t \left(\sqrt{V_t} dW_t^{(S)} + \lambda^{(S)} V_t dt \right) \\ & + \left\{ h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} \left(c_v - \widetilde{c}_v \right) \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} + \lambda^{(V)} \sqrt{V_t} dt \right) \\ & + \left\{ \left(\Delta h - \widetilde{h}_s \Delta S \right) - \left(\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S} \right) \right. \\ & \left. - \widetilde{\phi}_t^{(C)} \left[\left(\Delta c - \widetilde{h}_s \Delta S \right) - \left(\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S} \right) \right] \right\} dN_t. \end{aligned} \quad (10)$$

If in the true model, volatility is not stochastic, the term in the second line is set equal to zero, and if there is no jump risk, the term in the third line is set equal to zero. Analogously, the sensitivities are set equal to zero if the corresponding risk factor does not exist in the true model or in the hedge model, respectively.

The hedging error from Equation (10) can be decomposed into errors due to stock diffusion risk, volatility diffusion risk, and jump risk. Each of the three summands is equal to the remaining exposure to this risk factor, multiplied with the risk factor and its compensation. Consider the impact of volatility diffusion risk. The remaining exposure of the hedge portfolio – which is zero in case of a perfect hedge – can be explained by two kinds of errors. First, the sensitivities of the claim to be hedged are calculated in the hedge model and therefore deviate from the true partials $(h_v - \widetilde{h}_v)$. Second, the sensitivity of the hedge instrument C with respect to volatility is also calculated in the wrong model, so that the wrong number of units of the hedge instruments is employed to eliminate a given volatility risk exposure $(c_v - \widetilde{c}_v)$. The impact of stock diffusion risk and of jump risk is similar.

Model risk does not necessarily imply that there is a hedging error. It may also happen that the errors in the calculation of the exposure to be hedged are offset by the errors in the calculation of the exposure of the instrument used for hedging. If this is the case, the hedge is robust with respect to model risk, and one objective in the following will be to search for such a robust hedge. Intuitively, the conjecture is that the hedge will be the more robust the more similar the claim to be hedged and the hedge instrument are. For the variance contract which is exposed to stochastic volatility and to jump risk, there are two natural candidates for a robust hedge. The first is an ATM straddle (or ATM option) which is often considered as a an instrument that is mainly exposed to volatility risk. The second candidate is an OTM put, which mainly provides exposure to downward jumps in the stock price.

To assess the impact of model risk, the next subsections contain an analysis of the hedging error from Equation (10) over the a short time interval of length dt. We assume that there is a one-standard deviation shock in each of the diffusion terms, i.e. we assume that the stock price changes by $\sqrt{V_t} S_t dt$. In case of stochastic volatility, the same is done for the stochastic volatility component, which is shocked by $\sigma_V \sqrt{V_t} dt$. For this infinitesimal change, the hedging error follows from the SDE. It is then annualized by dividing by dt and expressed as a fraction of the current price of the claim to be hedged. For the jump component, we consider the hedging error if there is a jump, and again express it as a fraction of the current price of the claim.

4.3 Parameter Risk

We start our analysis by considering the case of parameter risk. Here, the assumption is that the investor uses the correct model (e.g., Heston (1993) or Merton (1976)), but with an incorrect parametrization. Even if an investor knew the true model type with certainty, he or she would have to estimate the parameters and thus could not avoid estimation risk (parameter risk).

4.3.1 Stochastic Volatility Model

In the model suggested by Heston (1993), the local variance of the stock follows a mean-reverting square-root process. The dynamics under the true measure are given by

$$\begin{aligned} dS_t &= (r + \lambda^S V_t)\, dt + \sqrt{V_t} S_t dW_t^S \\ dV_t &= \kappa^P(\theta^P - V_t)\, dt + \sigma_V \sqrt{V_t} \left(\rho dW_t^S + \sqrt{1-\rho^2} dW_t^V \right). \end{aligned}$$

We consider the hedging error under parameter risk for a hedge portfolio consisting of the stock, the hedge instrument and the money market account. The SDE for the hedging error D is stated in

Proposition 1 (SV under Parameter Risk)

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} (c_s - \widetilde{c}_s) \right\} (dS_t - rS_t dt) \\ &\quad + \left\{ h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} (c_v - \widetilde{c}_v) \right\} \left(dV_t - \kappa^Q(\theta^Q - V_t) dt \right), \end{aligned} \tag{11}$$

where the number of units of the hedge instrument is given by

$$\widetilde{\phi}_t^{(C)} = \frac{\widetilde{h}_v}{\widetilde{c}_v}.$$

A proof can be found in Appendix A.3.

The first term in brackets on the right hand side of Equation (11) is the interest earned on the hedging error accumulated up to time t. It is not relevant for our analysis, since it does not depend on the choice of the hedge portfolio at time t. We rather focus on those components of the local hedging error that could still be avoided if we knew the correct model, i.e. on the second and third term.

Despite parameter risk there is still a chance for the hedge to produce a zero error. When the errors in the sensitivities of the claim and the hedge instrument exactly offset each other, i.e. when

$$h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} (c_s - \widetilde{c}_s) = 0$$

and

$$h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} (c_v - \widetilde{c}_v) = 0,$$

the hedging error vanishes. These robustness conditions trivially hold in the true model. If the hedge model deviates from the true model, our conjecture is that the more similar the claim to be hedged is to the hedge instrument, the more similar the partial derivatives and also the associated errors will be, and thus the lower the replication error. This implies that when we use options as our hedge instruments, other options with a different strike should be easier to hedge than the variance contract. Furthermore, the hedge for a put should be the better the smaller the difference of the strike prices between the put and the option used in the hedge.

Figure 2 shows the local hedging errors. The parameters of the true model are taken from Bates (2000) (except for rounding), and the calibration of the hedge model was done as described in Section 4.2. As we can see from the left column of graphs, the hedge for the deep OTM-put with a moneyness of 0.85 is the better the lower the difference between the strike prices of the put to be hedged and the hedge instrument. Trivially, the 'ideal' hedge instrument is an option with identical strike. The important issue is that the two curves for the hedging errors generated by a one standard deviation shock in the state variables are still close to zero for strikes in the vicinity of 0.85. We can thus identify the best hedge instrument with a high degree of accuracy. For the variance contract (right column of graphs) the overall size of the relative hedging errors is the same as for the put. Nevertheless, model risk is a much more severe problem for the variance contract than for the put. First, there is no ideal hedge instrument for which the remaining exposure to both risk factors would vanish simultaneously. Second, and more importantly, it does not seem possible to determine a strike, i.e to choose a hedge instrument, for which the hedge is robust with respect to parameter risk. For the two different calibrated parameter sets, which fit the given prices of options with relative errors well below 1%, the optimal strike for the option used as the hedge instrument varies considerably. While in the upper graph, a strike between 90 and 105 gives a quite good hedge, the lower graph suggests that an option with a strike price of around 87 should be used for hedging, but that an ATM option results in quite large hedging error due to volatility risk. It is thus not possible to pick a 'best' hedge instrument, and an option that appears very good under one parameter set may perform poorly under another.

Finally, we are interested in the performance of the hedge if an ATM option is used as the hedge instrument. This choice is motivated by the intuitive idea that an ATM straddle is a good instrument to trade volatility. Looking at the graphs it becomes clear that the ATM option indeed provides an acceptable hedge for the variance contract under the first set of parameters, where hedging errors for a strike around 100 seem rather small. However, for the second parameter set this is no longer true, since especially variance shocks can cause considerable hedging errors. This finding confirms that the variance contract differs from its natural competitor ATM option.

4.3.2 Jump-Diffusion Model

Parameter risk can also be analyzed in the jump-diffusion (JD) model suggested by Merton (1976), with the slight variation that we assume a deterministic jump size. The SDE for the hedging error D is given in the next proposition:

Proposition 2 (JD under Parameter Risk)

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sigma S_t dW_t \\ &\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \left(\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S} \right) - \widetilde{\phi}_t^{(C)} \left[\Delta c - \widetilde{c}_s \Delta S - \left(\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S} \right) \right] \right\} dN_t \\ &\quad + \ldots dt \end{aligned} \tag{12}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the portfolio and where the number of claims is given by

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta\widetilde{h} - \widetilde{h}_s \Delta\widetilde{S}}{\Delta\widetilde{c} - \widetilde{c}_s \Delta\widetilde{S}}.$$

The proof can be found in Appendix A.4. Note that the jump size ΔS in the true model can well be different from the assumed jump size $\Delta\widetilde{S}$ in the hedge model. The interpretation of the SDE for the hedging error is similar to the case of the SV model discussed in the previous subsection. There are again robustness conditions under which the hedge will produce a zero error, despite the fact that incorrect parameters are used.

Figure 3 shows the relative hedging error for a local one standard deviation change $\sigma S_t dt$ in the stock price, and for the case when a jump occurs, i.e. for a change of the stock price by $\Delta S = \mu_X S_{t-}$. The results are in general similar to those found for the SV model. For the deep OTM put the hedge is the better the smaller the difference between its strike and the strike of the option used as a hedge instrument. The variance contract is again more difficult to hedge than the put. As in the SV case there is no choice of the hedge instrument for which the hedge would be insensitive to model risk. The ATM option, which is considered as a natural instrument to trade volatility risk, is again not the ideal hedge instrument. Furthermore, an OTM put which is often considered as a reliable hedge against jump risk does not perform well either.

The pictures also show that jump risk is in general more difficult to hedge than stock price risk or volatility risk. Note that this cannot be explained by a generic model incompleteness due to jumps or by the fact that a local delta hedge could possibly not control for large changes in the stock price due to a jump. Our model is complete by construction, so that a perfect hedge is basically feasible. The hedging errors are only caused by parameter risk, and the results show that especially the estimation of the jump size is a quite severe problem.

4.4 Mis-Specification of Risk Factors

Model mis-specification describes a situation where the wrong model is used for the hedge, and not just a model of the correct type with incorrect parameter values. In this section we focus on the case of a mis-specification of risk factors. The investor assumes the JD model, although the true model is SV, and vice versa. In Section 4.5 we will then analyze the case of omitted risk factors, where the hedge model is a restricted version of the true model. Intuitively, we would expect that model mis-specification has much more severe consequences for the general structure of hedging errors than an incorrect parametrization.

4.4.1 Stochastic Volatility Model

We start with the case when SV is the true model, but JD is used as the hedge model. Both models can, for example, produce a downward sloping smile. The investor might not be able to distinguish between SV and jumps, and might well decide to hedge jump risk while in fact stochastic volatility should be hedged. The following proposition gives the SDE for the hedging error D in the true (SV) model:

Proposition 3 (SV under Model Mis-Specification (Hedge: JD))

$$\begin{aligned} dD_t &= (H_t - \Pi_t)rdt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s\right) \right\} \sqrt{V_t} S_t dW_t^{(S)} \\ &\quad + \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) + \ldots dt \end{aligned} \tag{13}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and the number of units of the hedge instrument is given as

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}}.$$

The proof of the proposition is analogous to the one for Proposition 1 in Appendix A.3 and is omitted to save space.

The structure of the remaining exposure to diffusion risk of the stock (after the delta hedge) is well-known by now. A comparison with the case of estimation risk in Proposition 1 shows that model mis-specification and estimation risk differ in terms of the remaining exposure to volatility risk. When the wrong parameters are used, this remaining exposure depends on the error in the sensitivities with respect to volatility risk for both the claim and the hedge instrument. In Proposition 3, however, there is no stochastic volatility in the hedge model, and the sensitivities are set equal to zero. The error is not that sensitivities are computed *incorrectly*, but that this risk factor is *completely ignored*. Instead, the investor computes the additional exposure to jump risk in the hedge model and chooses the position in the hedge instrument to eliminate this exposure, which is not even present in the true model. While the position in the hedge instrument should depend on the ratio of the sensitivities with respect to volatility risk, it is calculated based on the assumed exposures to jump risk. Only if these two ratios coincide by chance the hedge will be correct.

Figure 4 shows the relative hedging errors for the OTM put and the variance contract. Again, these errors are based on parameter vectors for the hedge model which are compatible with observed market prices for a certain set of options. The general result is that hedging errors are much larger than under parameter risk, which confirms our intuition. This not only holds for volatility risk, which is hedged in a fundamentally wrong manner, but also for stock price risk.

A comparison of the contracts again shows that the OTM put is less exposed to model risk than the variance contract. The option can be hedged rather well using some other put with a similar strike, so that the rule for selecting its optimal hedge instrument still applies under model risk. Also similar to previous results we cannot find a robust hedge for the variance contract, as can be seen from the graphs in the figure. In particular, an OTM put with strike-to-spot ratio of 0.95, which might be recommended as a hedge against downward jump risk, performs quite well for the second parameter set, but rather poorly for the first. The ATM option, on the other hand, provides a pretty good hedge under the first calibrated parameter set, but not under the second.

4.4.2 Jump-Diffusion Model

Now the situation will be reversed, and JD with deterministic jump size will be the true model, while SV will be used as the hedge model. The SDE for the hedging error D under the true model is the content of the next proposition.

Proposition 4 (JD under Model Mis-Specification (Hedge: SV))

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sigma S_t dW_t \\ &\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \widetilde{\phi}_t^{(C)} \left(\Delta c - \widetilde{c}_s \Delta S \right) \right\} dN_t + \ldots dt, \end{aligned} \tag{14}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and where the number of claims is determined as

$$\widetilde{\phi}_t^{(C)} = \frac{\widetilde{h}_v}{\widetilde{c}_v}.$$

The proof of this proposition is analogous to that for Proposition 2 and is omitted to save space.

The structure of the remaining exposure to diffusion risk of the stock has already been discussed extensively. The remaining exposure to the jump risk of the stock is much more interesting. Like for the SV model the remaining exposure under model risk in Equation (14) is quite different from its counterpart under parameter risk in Equation (12). Under parameter risk the hedging error depends on the difference between the exposures to jump risk in the true and in the hedge model for both the claim and the hedge instrument. Here the hedge model does not even contain a jump component, so that the exposure with respect to this risk factor is basically set equal to zero. Instead the investor aims at hedging volatility risk not present in the true model. Again the hedge is only correct, if, by chance, the ratio of exposure to volatility in the hedge model is equal to the ratio of the exposures to additional jump risk in the true model.

Figure 5 shows the familiar result that the hedging error for the OTM put can be kept small by choosing a put with a similar strike price as the hedge instrument. Again, there is no ideal hedge instrument for the variance contract, which further underlines that this derivative asset is harder to replicate in a world with model risk than the simple put. Especially the impact of a jump is quite pronounced.

4.5 Model Risk: Missing Risk Factors

Another variant of model mis-specification is that the hedge model is less general than the true model, i.e. some of the risk factors included in the true model are omitted from the hedge model. For example, the true model could contain a multi-factor specification for stochastic volatility, as in Bates (2000), whereas the hedge model is a one-factor model like the one suggested by Heston (1993). It could also be the case that the general model developed by Bakshi, Cao, and Chen (1997) generates the data, while the hedge model is a restricted variant, like Heston (1993) or Merton (1976), where either stochastic jumps or stochastic volatility are missing. We will discuss two cases of omitted risk factors, where the true model is always given by a version of Bakshi, Cao, and Chen (1997) with a deterministic jump size for the stock. First, the jump component is omitted in the hedge model, and second, we analyze the consequences of leaving out the stochastic volatility part instead.

Again, this kind of model risk is quite likely to strike when a hedge is implemented. The true model will be able to explain many observable phenomena correctly, and it will usually be quite sophisticated with a large number of state variables and parameters. Even if we can find the correct type of model its parameters will be difficult to identify. We thus assume that the investor uses a simpler model which fits the data 'sufficiently' well. Once such a simpler model has been found, it would be hard to justify a more complex approach.

4.5.1 Omitted Jump Component

We start our analysis with the case where the jump component is missing from the hedge model. As usual, we first derive the dynamics of the hedging error in the true model:

Proposition 5 (BCC Model: Hedge under Model Mis-Specification (SV))

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sqrt{V_t} dW_t^{(S)} \\ &\quad + \left\{ h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} \left(c_v - \widetilde{c}_v \right) \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) \\ &\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \widetilde{\phi}_t^{(C)} \left(\Delta c - \widetilde{c}_s \Delta S \right) \right\} dN_t + \ldots dt \qquad (15) \end{aligned}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and where the number of claims is

$$\widetilde{\phi}_t^{(C)} = \frac{\widetilde{h}_v}{\widetilde{c}_v}.$$

The proof is similar to that for Proposition 1 in Appendix A.3.

The remaining exposure to stock price risk and to volatility risk has the same structure as in the case of parameter risk in the SV model discussed in Proposition 1. The most interesting part of Equation (15) is the one that relates to the jump risk exposure left in the hedge portfolio. As in Proposition 4, jump risk is missing from the hedge model and has been incorrectly interpreted as stochastic volatility. However, the investor now makes a second mistake. There are fewer risk factors in the hedge model than in the true model, so that there are not enough instruments in the hedge portfolio from the start, and the bad hedge for jump risk is unavoidable. Jump risk could now be eliminated only by chance, if the ratio of the additional exposure to jump risk for H and the hedge instrument coincides with the analogous ratio with respect to volatility risk.

Figure 6 shows the result of the analysis of the local relative hedging error. For the put the results are the same as in the cases studied before. One additional point to note is that the sensitivity of the hedge to shocks in the risk factors looks rather large even for this simple instrument, and that, depending on the model, volatility risk (present in the hedge model) may be as hard to hedge as jump risk (missing from the hedge model). Furthermore, looking at the left column of graphs we can see that the quality of the hedge deteriorates especially when the stock price has jumped. For the variance contract there is again no optimal hedge instrument which provides robustness with respect to model risk. Additionally, we have to keep in mind that the hedge model (Heston (1993)) is exposed to a kind of parameter risk. As discussed in Subsection 4.4.1, there may be more than one parametrization which fits the given prices, and these two problems add up in the hedging error.

4.5.2 Omitted Stochastic Volatility

As the last case we analyze the hedging error when the hedge model only contains a jump component, but no stochastic volatility.

Proposition 6 (BCC Model: Hedge under Model Mis-Specification (JD))

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sqrt{V_t} S_t dW_t^{(S)} \\ &\quad + \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) \\ &\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \left(\Delta\widetilde{h} - \widetilde{h}_s \Delta\widetilde{S} \right) - \widetilde{\phi}_t^{(C)} \left[\left(\Delta c - \widetilde{c}_s \Delta S \right) - \left(\Delta\widetilde{c} - \widetilde{c}_s \Delta\widetilde{S} \right) \right] \right\} dN_t \\ &\quad + \ldots dt \end{aligned} \tag{16}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and where the number of claims is

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta\widetilde{h} - \widetilde{h}_s \Delta\widetilde{S}}{\Delta\widetilde{c} - \widetilde{c}_s \Delta\widetilde{S}}.$$

The proof is analogous to that for Proposition 2 in Appendix A.4.

The interpretation of Equation (16) is very similar to those for the previous propositions. The structure of the remaining exposure to jump risk and to stock diffusion risk is identical to the case of parameter risk which has been given in Proposition 2. For volatility risk the structure of the remaining exposure had already been discussed in Proposition 4, where stochastic volatility was also not included in the hedge model. However, in the case where stochastic volatility had incorrectly been interpreted as jump risk, there would at least have been the chance to construct the correct hedge, since the right set of instruments was available. Here, this is basically impossible, since there are not enough instruments in the hedge portfolio to achieve completeness. Volatility risk is thus only hedged by chance.

Figure 7 compares the hedging errors for a deep OTM put and the variance contract. It shows that a hedge for the variance contract based on a mis-specified model can generate substantial hedging errors and that there is no robust choice of hedge instruments. For example, the upper graph in the right column seems to suggest that including a put with a strike price of roughly 106 generates relatively small errors. However, the lower graph shows that this does not hold in general. The hedging errors for this hedge instrument can be rather large under a different parameter scenario, which nevertheless prices the set of given contracts with acceptable precision. In particular, the remaining exposure to stochastic volatility is rather large.

5 Conclusion

Variance contracts are innovative derivative assets. They provide exposure to the variation risk of a stock, that is to stochastic volatility and jumps. Empirically there is evidence for a negative risk premium on the variance contract, which is usually explained by the well-documented negative market price of risk for stochastic volatility. A second explanation, however, is that stock price jumps and/or volatility jumps can be perceived to be more severe and more frequent under the risk-neutral than under the physical measure. The variance contract does not allow to distinguish between the risk factors volatility and jumps.

The main motivation for trading the variance contract comes from the optimal portfolio decision of investors. The investor wants an instrument providing him with exposure to variation risk of the stock, which has a high payoff if the diffusion variance has increased or if there have been jumps in the stock price. A formal motivation can be derived from the literature on portfolio planning, which shows that investors can have a demand for portfolios providing a hedge against fluctuations in the state variables. Given this (mainly speculative) demand, the question is then why investors would prefer the variance contract to a replicating strategy using standard options.

In our opinion the main economic motivation for the introduction of variance contracts is that the variance contract is 'better' than its replicating strategy and that it provides a more significant improvement relative to its replicating strategy than a standard option. This implies that there is a stronger motivation to introduce the variance contract than the option. There are at least two arguments for the superiority of the variance contract compared to a replication strategy. First, the semi-static hedging strategy fails if there are jumps in the stock price. Second, every dynamic replication strategy has to be based on some assumed model, and a hedging error will result if the hedge model is not equal to the true model.

In the paper, we consider the cases of parameter uncertainty and of mis-specified and omitted risk factors. Under all of these scenarios we derive analytical expressions for the local hedging errors. A graphical analysis shows that the hedging error for the variance contract is in general slightly larger than that for a deep OTM put, which was chosen as the benchmark asset and as an alternative candidate for the new derivative contract to be introduced. However, the variance contract is exposed to model risk to a much higher degree. For the put, the hedge is the more robust against model risk the smaller the difference between its strike price and the strike price of the hedge instrument. For the variance contract, there is no option for which the hedge is robust against model risk. Dynamic hedges for the variance contract are thus much riskier than those for put options. This model risk is the reason why most investors will prefer the variance contract to its supposedly replicating strategy. Furthermore, it causes a competition in terms of modeling and hedging competence between highly sophisticated investors providing liquidity in the variance contract.

A Appendix

A.1 Pricing of Variance Contract

The payoff of the variance contract at time T is

$$C_T = RV(0,T) = \int_0^T V_u du + \int_0^T X_u^2 dN_u.$$

Risk-neutral pricing then gives

$$\begin{aligned}
C_t &= E^Q\left[e^{-r(T-t)}\left(RV(0,t)+RV(t,T)\right)|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left\{RV(0,t)+\int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du+\int_t^T E^Q\left[X_u^2 dN_u|\mathcal{F}_t\right]\right\} \\
&= e^{-r(T-t)}\left\{RV(0,t)+\int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du+\int_t^T E^Q\left[E^Q\left[X_u^2|\mathcal{F}_{u-}\right]\left(k_0^Q+k_1^Q V_{u-}\right)|\mathcal{F}_t\right]du\right\} \\
&= e^{-r(T-t)}\left\{RV(0,t)+\int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du+\int_t^T E^Q\left[E^Q\left[X^2\right]\left(k_0^Q+k_1^Q V_{u-}\right)|\mathcal{F}_t\right]du\right\}
\end{aligned}$$

where the last equality follows from the assumption that the jump size X of the log return neither depends on time u nor on the other state variables. By rearranging the equation, we get

$$C_t = e^{-r(T-t)}\left\{RV(0,t)+\left(1+k_1^Q E^Q\left[X^2\right]\right)\int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du+(T-t)k_0^Q E^Q\left[X^2\right]\right\}.$$

To calculate the expectation of the variance, we start from the SDE

$$dV_t = \kappa^Q\left(\theta^Q-V_t\right)dt+\sigma_V\sqrt{V_t}\left(\rho d\widetilde{W}_t^{(1)}+\sqrt{1-\rho^2}d\widetilde{W}_t^{(2)}\right)+\left\{Y_t dN_t-\mu_Y\left(k_0^Q+k_1^Q V_t\right)dt\right\}.$$

Taking expectations gives

$$dE^Q\left[V_u|\mathcal{F}_t\right] = \kappa^Q\left(\theta^Q-E^Q\left[V_u|\mathcal{F}_t\right]\right)du.$$

The solution to the ordinary differential equation is

$$E^Q\left[V_u|\mathcal{F}_t\right] = e^{-\kappa^Q(u-t)}V_t+\left(1-e^{-\kappa^Q(u-t)}\right)\theta^Q.$$

Plugging this into the pricing equation, we get

$$\begin{aligned}
C_t &= e^{-r(T-t)}\left\{\int_0^t V_u du+\int_0^t X_u^2 dN_u+(T-t)k_0^Q E^Q\left[X^2\right]\right. \\
&\qquad \left.+\left(1+k_1^Q E^Q\left[X^2\right]\right)\int_t^T\left[e^{-\kappa^Q(u-t)}V_t+\left(1-e^{-\kappa^Q(u-t)}\right)\theta^Q\right]du\right\} \\
&= e^{-r(T-t)}\left\{\int_0^t V_u du+\int_0^t X_u^2 dN_u+(T-t)k_0^Q E^Q\left[X^2\right]\right. \\
&\qquad \left.+\left(1+k_1^Q E^Q\left[X^2\right]\right)\left(\frac{1-e^{-\kappa^Q(T-t)}}{\kappa^Q}\left(V_t-\theta^Q\right)+(T-t)\theta^Q\right)\right\}.
\end{aligned}$$

A.2 Expected Return of the Variance Contract

To calculate the expected return of the variance contract, we first derive the dynamics of the claim price. From the pricing equation for the variance contract, we get (after some simple, but time-consuming manipulations of the equations) the SDE

$$\begin{aligned} dC_t &= rC_t dt + e^{-r(T-t)} \Big\{ V_t dt + X_t^2 dN_t - E^Q[X^2] k_0^Q dt \\ &\quad - \left(1 + k_1^Q E^Q[X^2]\right) \left[e^{-\kappa^Q (T-t)} V_t + \left(1 - e^{-\kappa^Q (T-t)}\right) \theta^Q \right] dt \\ &\quad + \left(1 + k_1^Q E^Q[X^2]\right) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \, dV_t \Big\}. \end{aligned}$$

To obtain the risk premium, we compare the drift of the price under the physical measure P and under the risk-neutral measure Q. With a slight abuse of notation, we get

$$\begin{aligned} & E^P\left[dC_t | \mathcal{F}_t\right] - E^Q\left[dC_t | \mathcal{F}_t\right] \\ &= e^{-r(T-t)} \Big\{ E^P\left[X^2\right] k_t^P - E^Q\left[X^2\right] k_t^Q \\ &\quad + \left(1 + k_1^Q E^Q[X^2]\right) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \left(\kappa^P \left(\theta^P - V_t\right) - \kappa^Q \left(\theta^Q - V_t\right) \right) \Big\} dt \\ &= e^{-r(T-t)} \Big\{ E^P\left[X^2\right] k_t^P - E^Q\left[X^2\right] k_t^Q \\ &\quad + \left(1 + k_1^Q E^Q[X^2]\right) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \left(\lambda^V \sigma_V V_t + E^P[Y_t] k_t^P - E^Q[Y_t] k_t^Q \right) \Big\} dt. \end{aligned}$$

A.3 Proof of Proposition 1

From the definition of the hedging error, we know that

$$dD_t = dH_t - d\Pi_t.$$

For the claim price $H = h(t, S_t, V_t, \ldots)$, we can derive the SDE by using Ito

$$dH_t = h_t dt + h_s dS_t + h_v dV_t + \frac{1}{2} h_{ss} V_t S_t^2 dt + \frac{1}{2} h_{vv} \sigma_V^2 V_t dt + h_{sv} \rho \sigma_V V_t S_t dt.$$

Applying the fundamental partial differential equation then gives

$$dH_t = H_t r dt + h_s (dS_t - r S_t dt) + h_v \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right].$$

The same equation holds for the price of the claim C.

The hedge portfolio consists of $\widetilde{\phi}_t^{(S)}$ units of the stock, $\widetilde{\phi}_t^{(C)}$ units of the claim C, and an investment of $\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t^{(1)}$ in the money market account, which is chosen such that the portfolio is self-financing. The SDE for the value of the hedge portfolio is then

$$\begin{aligned} d\Pi_t &= \widetilde{\phi}_t^{(S)} dS_t + \widetilde{\phi}_t^{(C)} dC_t + \left(\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t\right) rdt \\ &= \Pi_t rdt + \widetilde{\phi}_t^{(S)} (dS_t - rS_t dt) + \widetilde{\phi}_t^{(C)} (dC_t - rC_t dt). \end{aligned}$$

Plugging the expressions for dH_t and $d\Pi_t$ into the definition of dD_t and sorting the terms by the risk factors, that is by stock price risk and volatility risk, gives

$$\begin{aligned} dD_t &= H_t rdt + h_s(dS_t - rS_t dt) + h_v \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right] \\ &\quad - \Pi_t rdt - \widetilde{\phi}_t^{(S)} (dS_t - rS_t dt) - \widetilde{\phi}_t^{(C)} (dC_t - rC_t dt) \\ &= H_t rdt + h_s(dS_t - rS_t dt) + h_v \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right] \\ &\quad - \Pi_t rdt - \widetilde{\phi}_t^{(S)} (dS_t - rS_t dt) - \widetilde{\phi}_t^{(C)} \left\{c_s(dS_t - rS_t dt) + c_v \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right]\right\} \\ &= (H_t - \Pi_t) rdt + \left\{h_s - \widetilde{\phi}_t^{(S)} - \widetilde{\phi}_t^{(C)} c_s\right\} (dS_t - rS_t dt) \\ &\quad + \left\{h_v - \widetilde{\phi}_t^{(C)} c_v\right\} \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right]. \end{aligned} \tag{17}$$

The number of claims in the hedge portfolio follows from the conditions

$$\begin{aligned} \widetilde{\phi}_t^{(S)} + \widetilde{\phi}_t^{(C)} \widetilde{c}_s &= \widetilde{h}_s \\ \widetilde{\phi}_t^{(C)} \widetilde{c}_v &= \widetilde{h}_v \end{aligned}$$

which yield

$$\begin{aligned} \widetilde{\phi}_t^{(S)} &= \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s \\ \widetilde{\phi}_t^{(C)} &= \frac{\widetilde{h}_v}{\widetilde{c}_v}. \end{aligned}$$

Plugging the number of stocks $\widetilde{\phi}_t^{(S)}$ into Equation (17) gives

$$\begin{aligned} dD_t &= (H_t - \Pi_t) rdt + \left\{h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s\right\} (dS_t - rS_t dt) \\ &\quad + \left\{h_v - \widetilde{\phi}_t^{(C)} c_v\right\} \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right] \\ &= (H_t - \Pi_t) rdt + \left\{h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left[c_s - \widetilde{c}_s\right]\right\} (dS_t - rS_t dt) \\ &\quad + \left\{h_v - \widetilde{\phi}_t^{(C)} c_v\right\} \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right] \\ &= (H_t - \Pi_t) rdt + \left\{h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left[c_s - \widetilde{c}_s\right]\right\} (dS_t - rS_t dt) \\ &\quad + \left\{h_v - \widetilde{h}_v + \widetilde{\phi}_t^{(C)} \widetilde{c}_v - \widetilde{\phi}_t^{(C)} c_v\right\} \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right] \\ &= (H_t - \Pi_t) rdt + \left\{h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left[c_s - \widetilde{c}_s\right]\right\} (dS_t - rS_t dt) \\ &\quad + \left\{h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} \left[c_v - \widetilde{c}_v\right]\right\} \left[dV_t - \kappa^Q(\theta^Q - V_t)dt\right]. \end{aligned}$$

A.4 Proof of Proposition 2

In the model of Merton (1976), the SDE for the stock price is

$$dS_t = rS_t dt + \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) + S_{t-}\left[\left(e^{X_t}-1\right)dN_t - E^Q[e^X-1]k_t^Q dt\right].$$

Again, we first derive the SDE for the price $H_t = h(t, S_t, \ldots)$ of a contingent claim. From Ito, we get

$$\begin{aligned} dH_t &= h_t dt + h_s\left\{dS_t - \left(e^{X_t}-1\right)S_{t-}dN_t\right\} + \frac{1}{2}h_{ss}\sigma^2 S_t^2 dt \\ &\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t. \end{aligned}$$

The price of the claim has to fulfill the fundamental partial differential equation, so that we get

$$\begin{aligned} dH_t &= rH_t dt + h_s\left\{dS_t - rS_t dt - \left(e^{X_t}-1\right)S_{t-}dN_t + E^Q[e^X-1]k_t^Q S_{t-}dt\right\} \\ &\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t - E^Q\left[h(t, se^X, \ldots) - h(t, s, \ldots)\right] k_t^Q dt \\ &= rH_t dt + h_s \sigma S_t\left(dW_t^{(S)} + \lambda^{(S)}dt\right) \\ &\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t - E^Q\left[h(t, se^X, \ldots) - h(t, s, \ldots)\right] k^Q dt. \end{aligned}$$

The same equation holds for the price of the claim C.

The hedge portfolio consists of $\widetilde{\phi}_t^{(S)}$ units of the stock, $\widetilde{\phi}_t^{(C)}$ units of the claim C, and an investment of $\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t$ in the money market account, which is chosen such that the portfolio is self-financing. The SDE for the value of the hedge portfolio is then

$$\begin{aligned} d\Pi_t &= \widetilde{\phi}_t^{(S)} dS_t + \widetilde{\phi}_t^{(C)} dC_t + \left(\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t\right) r dt \\ &= \Pi_t r dt + \widetilde{\phi}_t^{(S)}(dS_t - rS_t dt) + \widetilde{\phi}_t^{(C)}(dC_t - rC_t dt). \end{aligned}$$

The number of stocks in the hedge portfolio follows from

$$\widetilde{\phi}_t^{(S)} + \widetilde{\phi}_t^{(C)}\widetilde{c}_s = \widetilde{h}_s$$

so that

$$\widetilde{\phi}_t^{(S)} = \widetilde{h}_s - \widetilde{\phi}_t^{(C)}\widetilde{c}_s.$$

Plugging this expression for the number of stocks and the SDE for the claim price C into the SDE for the value of the hedge portfolio, the latter becomes

$$\begin{aligned} d\Pi_t &= \Pi_t r dt + \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)}\widetilde{c}_s\right\}(dS_t - rS_t dt) + \widetilde{\phi}_t^{(C)}(dC_t - rC_t dt) \\ &= \Pi_t r dt + \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)}\widetilde{c}_s + \widetilde{\phi}_t^{(C)} c_s\right\}\sigma S_t\left(dW_t^{(S)} + \lambda^{(S)}dt\right) \\ &\quad + \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)}\widetilde{c}_s\right\}\left[\left(e^{X_t}-1\right)S_{t-}dN_t - E^Q[e^X-1]S_{t-}k^Q dt\right] \\ &\quad + \widetilde{\phi}_t^{(C)}\left\{\left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] dN_t - E^Q\left[c(t, se^X, \ldots) - c(t, s, \ldots)\right] k_t^Q dt\right\}. \end{aligned}$$

Now we plug the SDEs for H and Π into the definition of dD_t:

$$\begin{aligned}
dD_t &= rH_t dt + h_s \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\
&\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t - E^Q \left[h(t, se^{X}, \ldots) - h(t, s, \ldots)\right] k_t^Q dt \\
&\quad - \Pi_t r dt - \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s + \widetilde{\phi}_t^{(C)} c_s\right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\
&\quad - \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s\right\} \left[S_{t-}\left(e^{X_t} - 1\right) dN_t - S_{t-} E^Q[e^{X_t} - 1] k_t^Q dt\right] \\
&\quad - \widetilde{\phi}_t^{(C)} \Big\{ \left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] dN_t - E^Q \left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] k_t^Q dt \Big\} \\
&= (H_t - \Pi_t) r dt + \left\{h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s\right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\
&\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t - E^Q \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] k_t^Q dt \\
&\qquad\qquad - \widetilde{h}_s \left[S_{t-}\left(e^{X_t} - 1\right) dN_t - S_{t-} E^Q[e^{X_t} - 1] k_t^Q dt\right] \\
&\quad - \widetilde{\phi}_t^{(C)} \Big\{ \left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] dN_t - E^Q \left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] k_t^Q dt \\
&\qquad\qquad - \widetilde{c}_s \left[S_{t-}\left(e^{X_t} - 1\right) dN_t - S_{t-} E^Q[e^{X_t} - 1] k_t^Q dt\right] \Big\}.
\end{aligned}$$

If the jump size is deterministic, then $e^{X_t} - 1 = \mu_X$, and the SDE for the hedging error becomes

$$\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s\right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\
&\quad + \left[h(t, s(1+\mu_X), \ldots) - h(t, s, \ldots)\right] dN_t - \left[h(t, s(1+\mu_X), \ldots) - h(t, s, \ldots)\right] k_t^Q dt \\
&\qquad\qquad - \widetilde{h}_s \left[S_{t-}\mu_X dN_t - S_{t-}\mu_X k_t^Q dt\right] \\
&\quad - \widetilde{\phi}_t^{(C)} \Big\{ \left[c(t, s(1+\mu_X), \ldots) - c(t, s, \ldots)\right] dN_t - \left[c(t, s(1+\mu_X), \ldots) - c(t, s, \ldots)\right] k_t^Q dt \\
&\qquad\qquad - \widetilde{c}_s \left[S_{t-}\mu_X dN_t - S_{t-}\mu_X k_t^Q dt\right] \Big\}.
\end{aligned}$$

With the abbreviations

$$\begin{aligned}
\Delta S &= S_{t-}\mu_X \\
\Delta \widetilde{S} &= S_{t-}\widetilde{\mu}_X \\
\Delta h &= h(t, s(1+\mu_X), \ldots) - h(t, s, \ldots) \\
\Delta \widetilde{h} &= \widetilde{h}(t, s(1+\widetilde{\mu}_X), \ldots) - \widetilde{h}(t, s, \ldots)
\end{aligned}$$

and the analogous terms for the claim C, we can write the SDE for the hedging error as

$$\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \Delta h \, dN_t - \Delta h \, k_t^Q dt - \widetilde{h}_s \left[\Delta S \, dN_t - \Delta S \, k_t^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \Delta c \; dN_t - \Delta c \; k_t^Q dt - \widetilde{c}_s \left[\Delta S \, dN_t - \Delta S \, k_t^Q dt \right] \right\} \\
&= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \Delta h \, dN_t - \Delta h \, k_t^Q dt - \widetilde{h}_s \left[\Delta S \, dN_t - \Delta S \, k_t^Q dt \right] \\
&\quad - \Delta \widetilde{h} \, dN_t + \Delta \widetilde{h} \, k_t^Q dt + \widetilde{h}_s \left[\Delta \widetilde{S} \, dN_t - \Delta \widetilde{S} \, k_t^Q dt \right] \\
&\quad + \Delta \widetilde{h} \, dN_t - \Delta \widetilde{h} \, k_t^Q dt - \widetilde{h}_s \left[\Delta \widetilde{S} \, dN_t - \Delta \widetilde{S} \, k_t^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \Delta c \; dN_t - \Delta c \; k_t^Q dt - \widetilde{c}_s \left[\Delta S \, dN_t - \Delta S \, k_t^Q dt \right] \right\}.
\end{aligned}$$

From the conditions on the hedge ratios, we know that

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}}$$

With this expression for the hedge ratio $\widetilde{\phi}_t^{(C)}$, we can rewrite the SDE as

$$\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \Delta h \, dN_t - \Delta h \, k^Q dt - \widetilde{h}_s \left[\Delta S \, dN_t - \Delta S \, k_t^Q dt \right] \\
&\qquad\qquad - \Delta \widetilde{h} \, dN_t + \Delta \widetilde{h} \, k_t^Q dt + \widetilde{h}_s \left[\Delta \widetilde{S} \, dN_t - \Delta \widetilde{S} \, k_t^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \Delta c \; dN_t - \Delta c \; k_t^Q dt - \widetilde{c}_s \left[\Delta S \, dN_t - \Delta S \, k_t^Q dt \right] \right. \\
&\qquad\qquad \left. - \Delta \widetilde{c} \; dN_t + \Delta \widetilde{c} \; k_t^Q dt + \widetilde{c}_s \left[\Delta \widetilde{S} \, dN_t - \Delta \widetilde{S} \, k_t^Q dt \right] \right\}.
\end{aligned}$$

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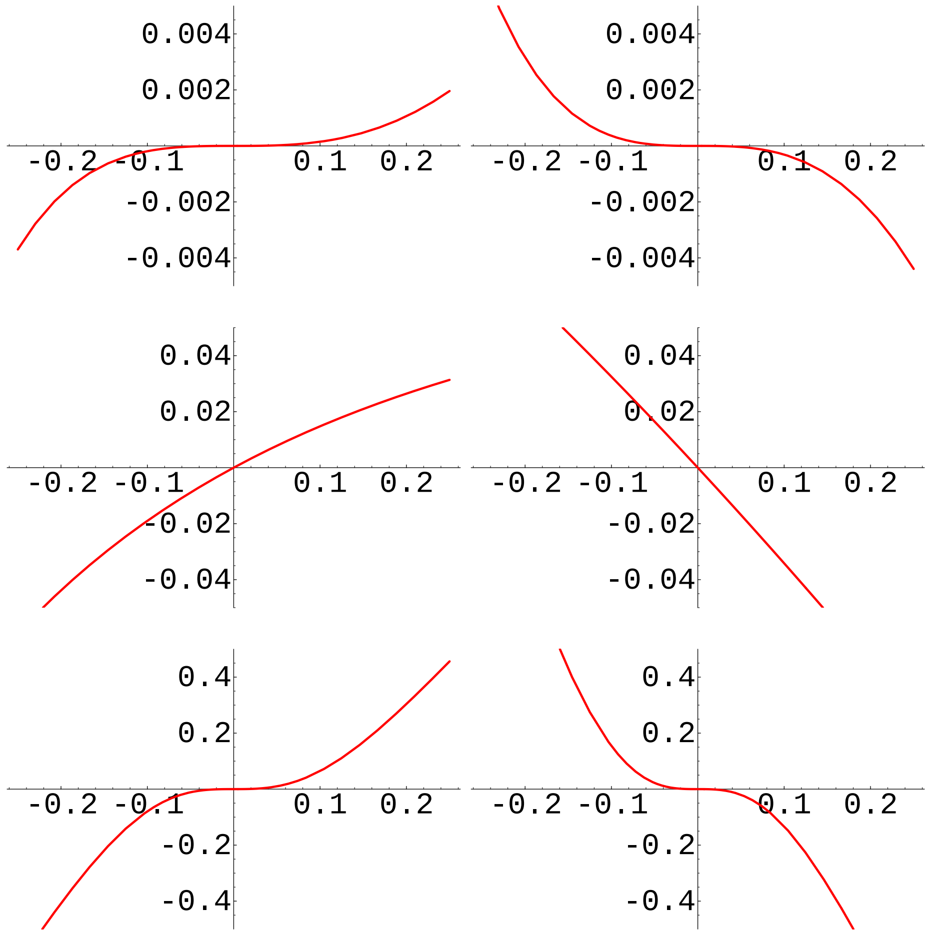

Figure 1: Hedging error of the 'semi-static replicating strategy' due to a jump

The figure gives the replication error of the 'replicating strategy' as a function of the relative jump size in the stock. If the error is negative, the payoff of the 'replicating strategy' is smaller than the payoff of the variance contract.

The left panel shows the error if the realized variance is based on log returns, and the right panel gives the error if it is based on discrete returns. The upper graphs give the size of the error, the middle graphs relate it to the payoff due to a jump, and the lower graphs express it as a function of the initial price of the contract (for a time to maturity of one month).

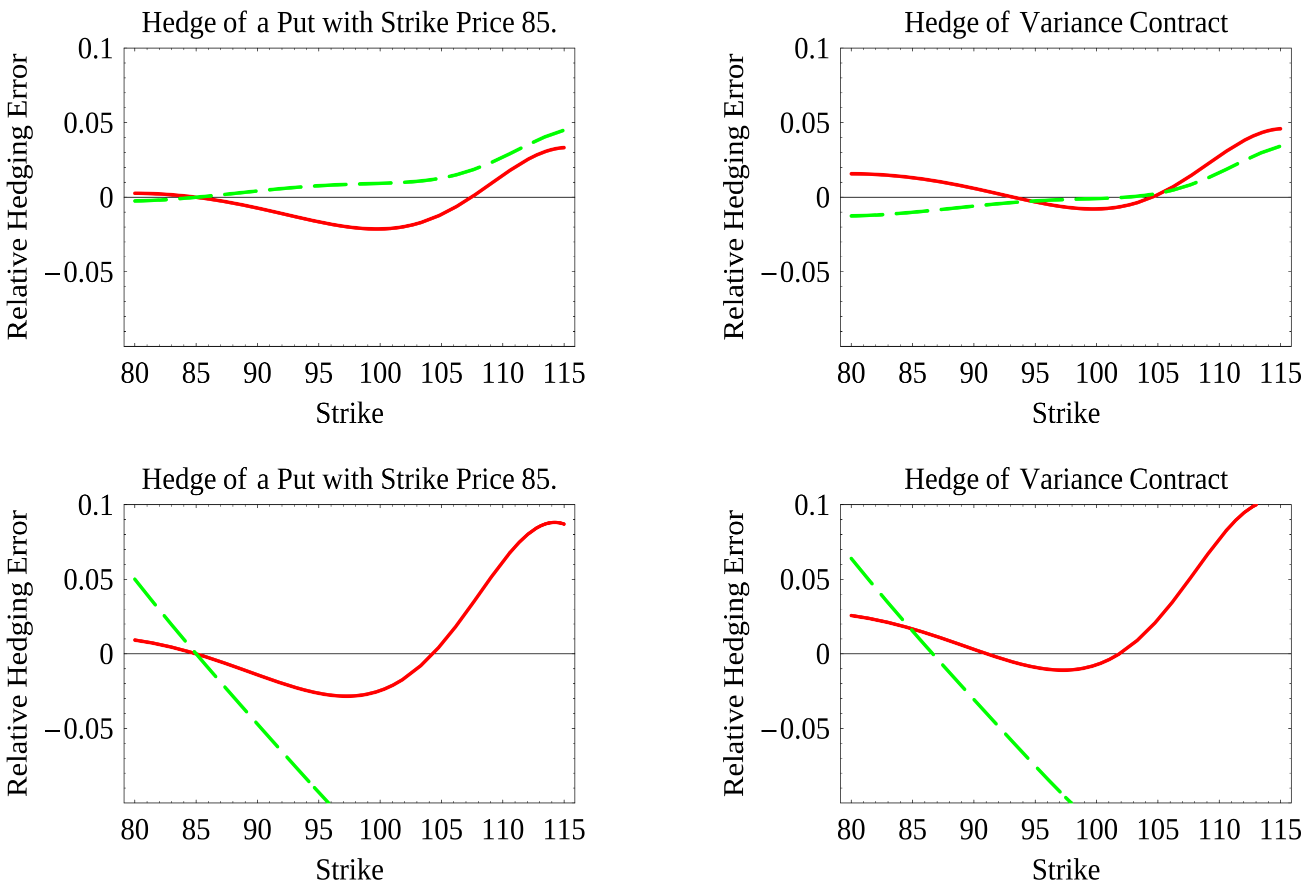

Figure 2: SV Model: Relative Hedging Error under Parameter Risk

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a change in squared volatility V by one standard deviation (dashed line).

The figures in the two rows are based on two different sets of parameters which price options with moneyness K/S of 0.90, 0.95, 1.0 and 1.05 with a relative error of less than 0.3%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

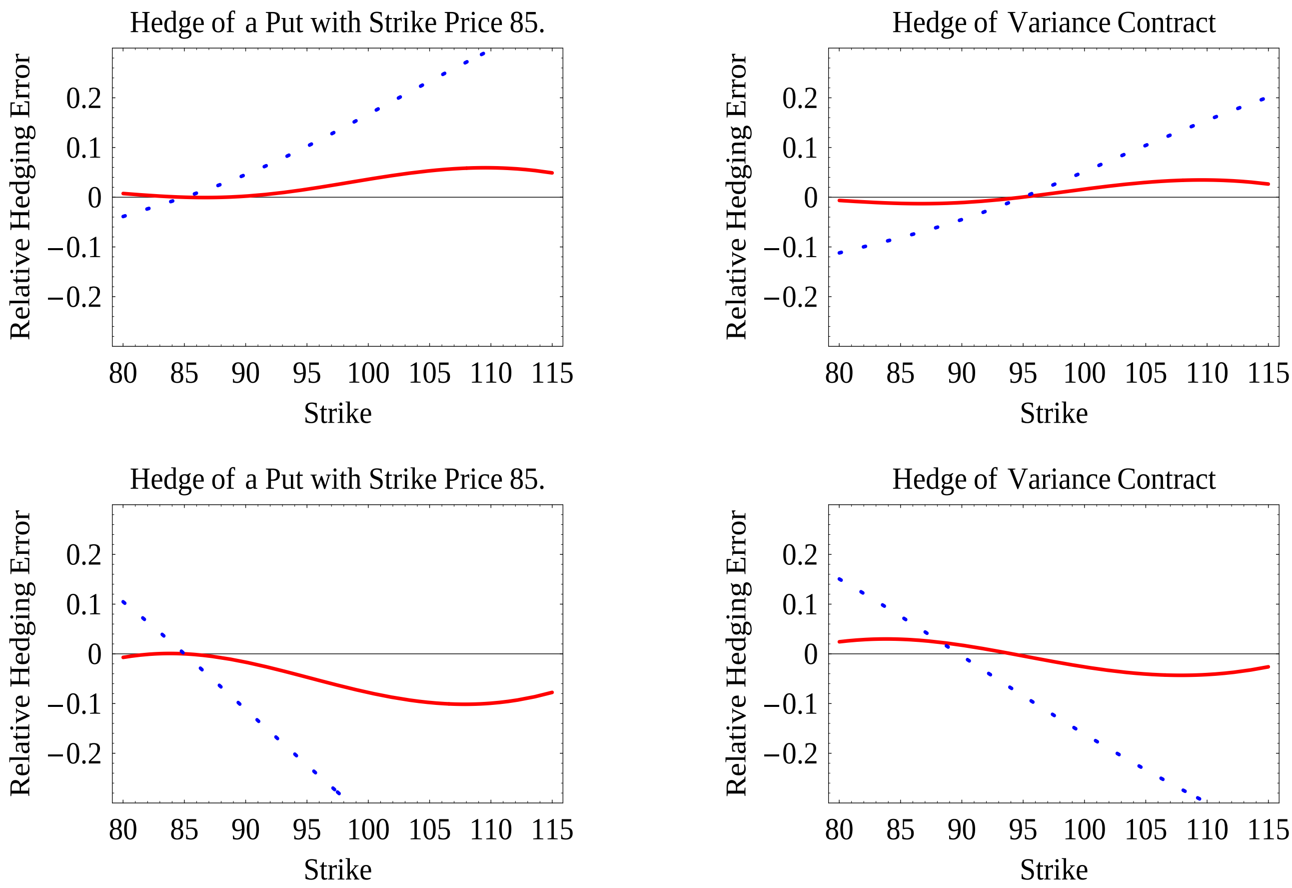

Figure 3: JD Model: Relative Hedging Error under Parameter Risk

The local relative hedging errors for a deep OTM put and for the variance quadratic contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two different hedge models which price options with moneyness K/S of 0.95, 1.0, and 1.05 with a relative error of less than 0.5%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

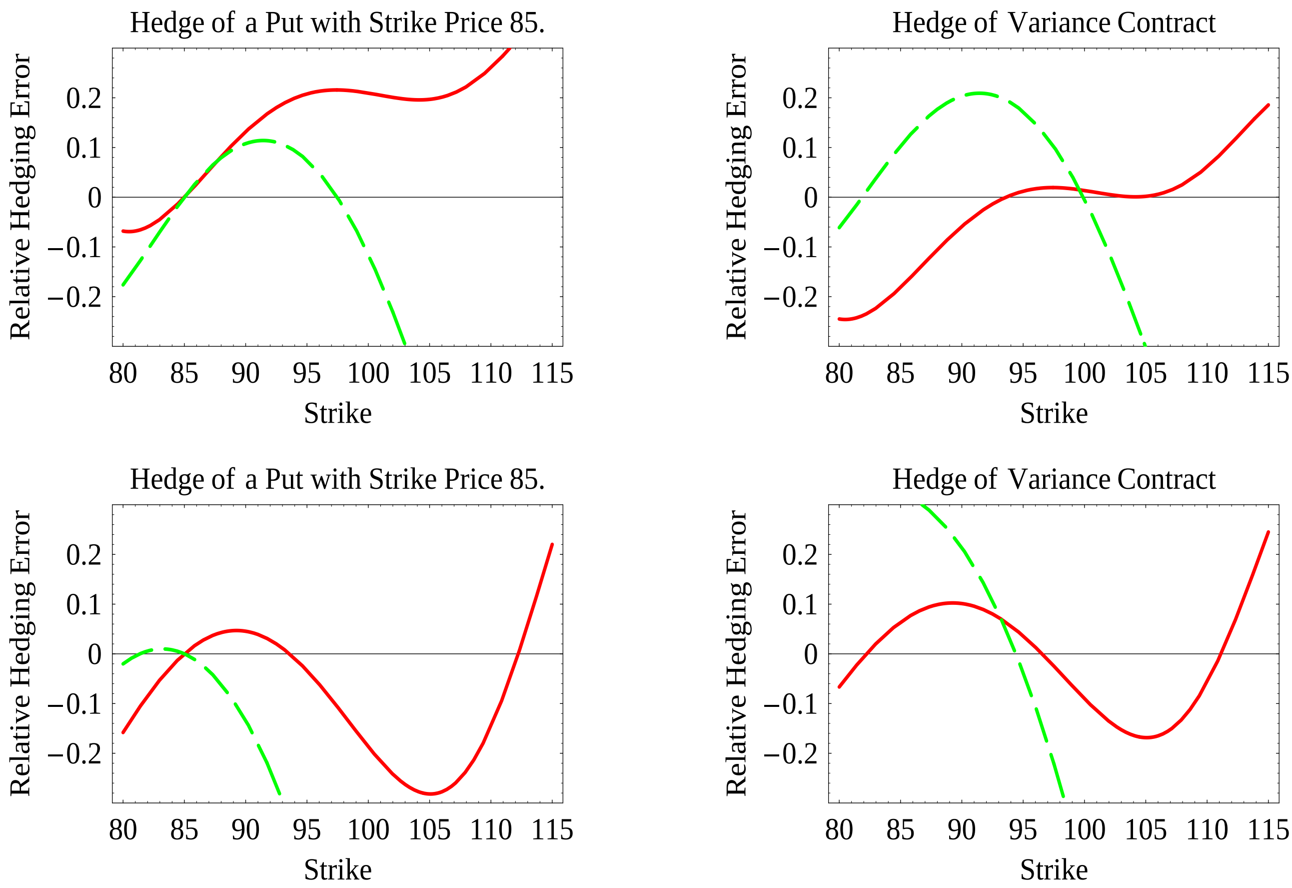

Figure 4: SV Model: Relative Hedging Error under Model Risk (JD)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a change in squared volatility V by one standard deviation (dashed line).

The figures in the two rows are based on two different sets of parameters which price options with moneyness K/S of 0.95, 1.0 and 1.05 with a relative error of less than 1.0%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

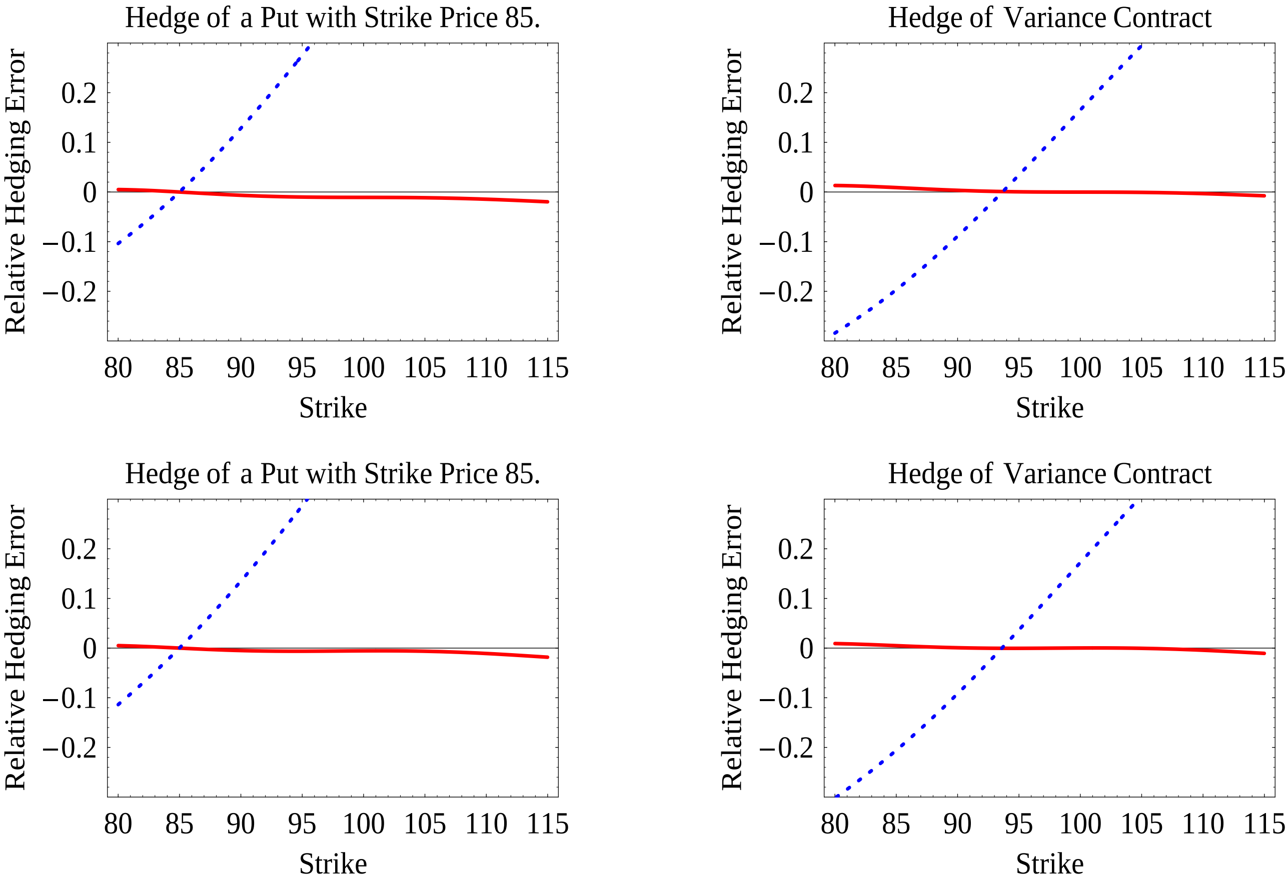

Figure 5: JD Model: Relative Hedging Error under Model Risk (SV)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two different parameter sets which price options with moneyness K/S of 0.90, 0.95, 1.0 and 1.05 with a relative error of less than 0.1%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

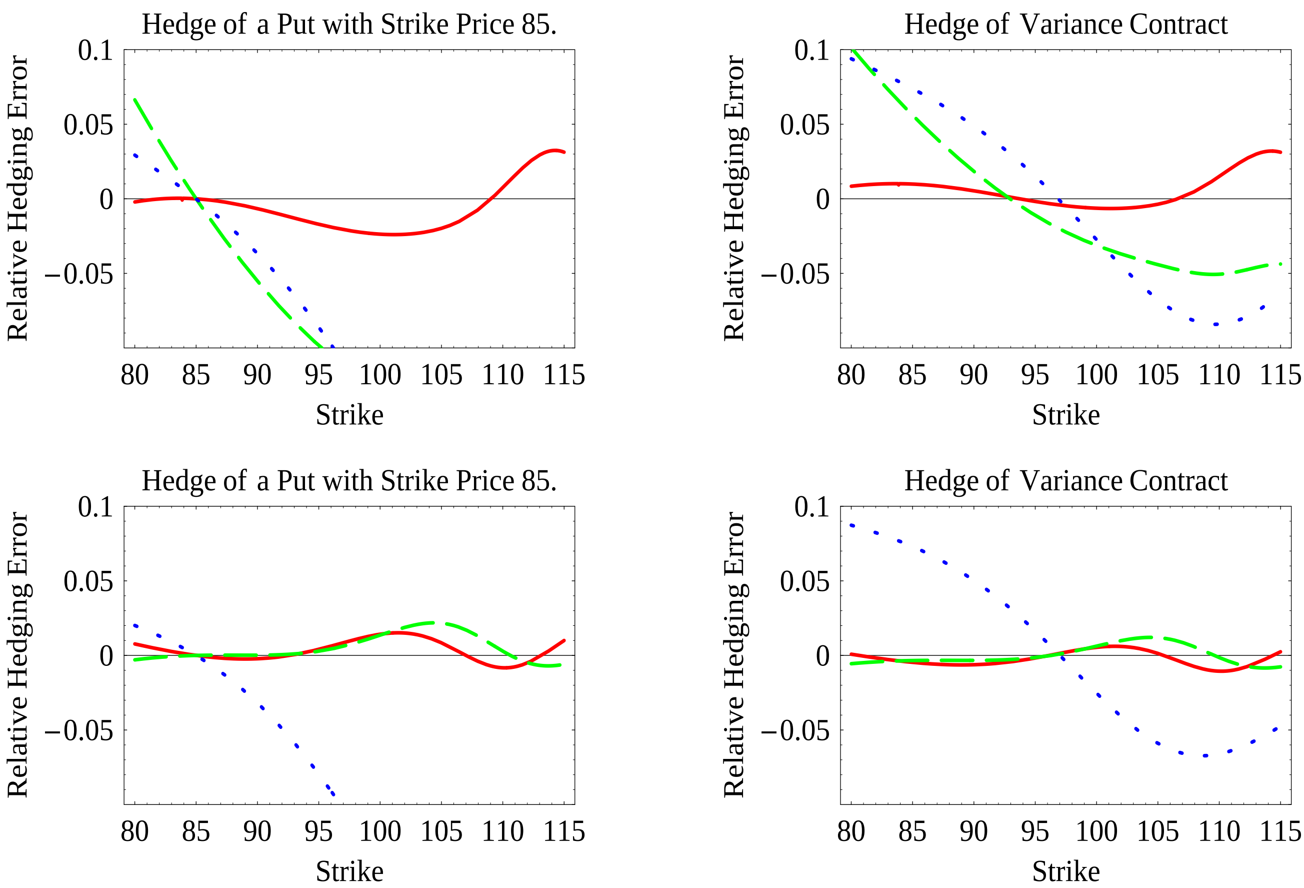

Figure 6: BCC Model: Relative Hedging Error under Model Risk (SV)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line), for a change in squared volatility V by one standard deviation (dashed line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two different sets of parameters which price options with moneyness K/S of 0.90, 0.95, 1.0 and 1.05 with a relative error of less than 0.3%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

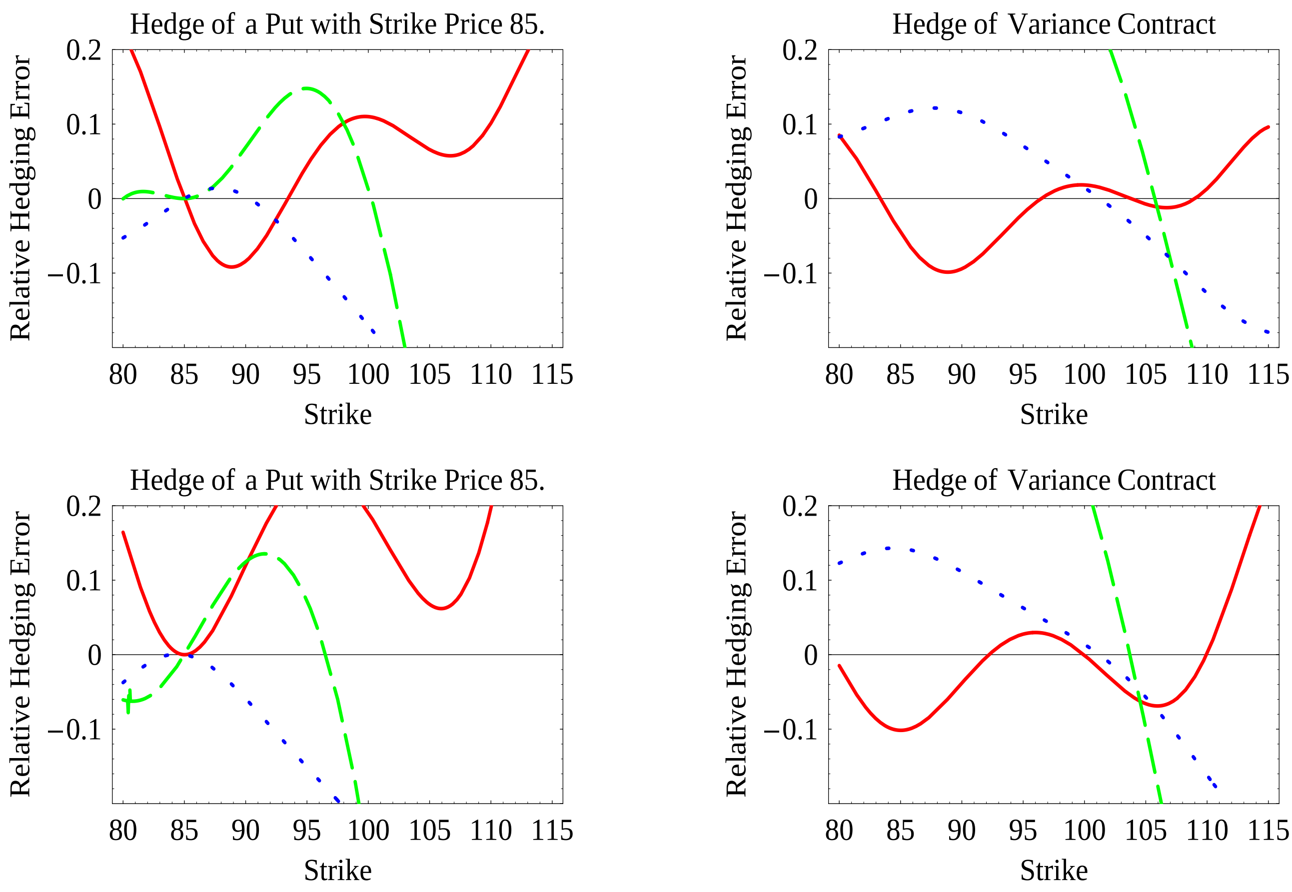

Figure 7: BCC Model: Relative Hedging Error under Model Risk (JD)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line), for a change in squared volatility V by one standard deviation (dashed line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two different sets of parameters which price options with moneyness K/S of 0.95, 1.0 and 1.05 with a relative error of less than 1.0%. The current stock price is equal to 100, the time to maturity of all contracts is six months.