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# A new stock-price bubble with stochastically deflating trajectories

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# A new stock-price bubble with stochastically deflating trajectories

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**Abstract**

We propose a new, rational stock-price bubble that is able to generate recurringly explosive and stochastically deflating trajectories. Our flexible bubble process entails stock-price volatility dynamics that are consistent with real-world data. To demonstrate this, we fit our bubble specification to NASDAQ data and analyze the volatility dynamics.

*Keywords:* Present-value model, Evans bubble, incompletely bursting bubble, stock-price volatility, particle-filter estimation.

*JEL classification:* C1, G1.

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# 1 Introduction

Evans (1991) specifies a rational, periodically collapsing stock-price bubble, which has become a benchmark model in the theoretical and empirical literature. However, two features inherent in Evans' specification appear to be irreconcilable with real-world data. (i) The Evans bubble always collapses entirely within one period, implying that stock-price volatility also collapses within one period (Rotermann and Wilfling, 2014). (ii) Whenever the Evans bubble bursts, it recedes to the same expected (non-zero) value, a property that is unnecessarily restrictive.

Figure 1 about here

Allen and Gale (2000) and Kindleberger and Aliber (2005), *inter alia*, provide a rationale as to why, after a crash, stock-price adjustment to a fundamentally justified level (i.e. the deflation of the bubble) typically follows a long and convoluted process. Corroborating empirical evidence is presented in Figure 1, which displays the monthly NASDAQ stock-market index between January 1990 and October 2013. The two shaded areas represent the deflation processes, starting with (i) the bursting dot-com bubble in March 2000, and (ii) the crash in the aftermath of the subprime mortgage crisis in October 2007. Typical of most stock-price deflation processes is that they are caused by bad news reaching investors in a highly uncertain financial-market environment, in which panic trading reactions accelerate the downward adjustment process. Such extreme stock-price downturns are occasionally interrupted by policy and/or regulatory interventions, which provide room for short-term price recovery.

In this paper, we establish a flexible bubble specification that overcomes the above-mentioned drawbacks of the Evans bubble. Our specification is consistent with rational expectations and generates recurringly explosive trajectories with stochastically deflating adjustments, and implies realistic stock-price volatility paths. Overall, our contributions are threefold. (i) We establish the stochastic properties of our bubble specification. (ii) We elaborate the impact of our bubble process on stock-price volatility. (iii) We fit our bubble specification to the NASDAQ data shown in Figure 1 and analyze the volatility implications.

## 2 Previous rational bubbles

In the linear present-value model with rational expectations, the price of a stock at date $t$, $P_t$, is given by the Euler equation

$$P_t = \frac{1}{1+r}\left[E_t(P_{t+1}) + E_t(D_{t+1})\right], \tag{1}$$

where $D_{t+1}$ is the stock dividend payment between $t$ and $t+1$. $E_t(\cdot)$ denotes the conditional expectation operator, based on all information available to market participants as of date $t$. $r$ is the *required rate of return* that is just sufficient to compensate investors for the inherent riskiness of the stock. The expectational difference equation (1) can be routinely solved by substituting future prices forward repeatedly. Denoting the stream of discounted expected future dividends by $P_t^f$ (the fundamental stock price) and the bubble component by $B_t$, we obtain the following familiar present-value formula for the stock-price at date $t$:

$$P_t = P_t^f + B_t = \sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i \cdot E_t(D_{t+i}) + B_t. \tag{2}$$

The entire class of solutions to the Euler Eq. (1) is given by Eq. (2), in which $B_t$ is any random variable satisfying the (discounted) martingale property

$$E_t(B_{t+1}) = (1+r)\cdot B_t \qquad \text{or, equivalently,} \qquad B_t = \frac{1}{1+r}\cdot E_t(B_{t+1}). \tag{3}$$

$B_t$ is called a rational bubble, because its presence in Eq. (2) is consistent with rational expectations.

Blanchard (1979) and Blanchard and Watson (1982) propose an early bubble specification that satisfies the martingale property (3). However, their specification is not necessarily consistent with two theoretical properties of rational stock-price bubbles. In view of Eq. (3), Diba and Grossman (1988) argue that, in general, (i) rational bubbles cannot start from zero, and (ii) negative bubbles are ruled out as $t \to \infty$.

Evans (1991) overcomes Diba and Grossman's fundamental critique by introducing his renowned rational bubble

$$B_t = \begin{cases} \frac{1}{\psi} B_{t-1} u_t & , \text{ if } B_{t-1} \leq \tau \\ [\kappa + \frac{1}{\pi\psi}\Big(B_{t-1} - \kappa\psi\Big)\nu_t] u_t & , \text{ if } B_{t-1} > \tau \end{cases} , \tag{4}$$

where $\psi \equiv (1+r)^{-1}$, $\kappa$ and $\tau$ are real constants to be chosen such that $0 < \kappa < (1+r)\tau$, and $\{u_t\}_{t=1}^{\infty}$ is an exogenous process of i.i.d. random variables with $u_t > 0$ and $E_{t-1}(u_t) = 1$ for all $t$. The variables $\{u_t\}$ are assumed to be lognormally (LN) distributed and scaled to have unit means, i.e. we assume $u_t = \exp(y_t - \iota^2/2)$ with $\{y_t\}_{t=1}^{\infty}$ being i.i.d. $N(0, \iota^2)$.[1] $\{\nu_t\}_{t=1}^{\infty}$ constitutes an exogenous i.i.d. Bernoulli process independent of $\{u_t\}_{t=1}^{\infty}$ with $\Pr(\nu_t = 1) = \pi$ and $\Pr(\nu_t = 0) = 1 - \pi$ for $0 < \pi \leq 1$. The event $\{\nu_t = 1\}$ signifies that the bubble will continue to grow, whereas the bubble bursts in the case of $\{\nu_t = 0\}$.

We note that the Evans bubble (4) features two different rates of growth. For $B_{t-1} \leq \tau$, the bubble grows at the mean rate $\frac{1}{\psi} - 1 = r$. For $B_{t-1} > \tau$, the bubble grows at the faster rate $\frac{1}{\pi\psi} - 1 > r$ (whenever $\pi < 1$), but collapses with probability $1 - \pi$ per period. As mentioned above, the Evans bubble (i) collapses entirely within one period, and (ii) necessarily returns to the same (positive) expected level $\kappa$ after bursting, from where the process recommences.

[1] In other words, $\{u_t\}$ represents an i.i.d. lognormal process with $u_t \sim \mathrm{LN}(\frac{-\iota^2}{2}, \iota^2)$.

An alternative process, providing a more flexible deflating behavior than the Evans bubble (4), is the incompletely bursting one proposed by Fukuta (1998), which consists of three potential states:

$$B_t = \begin{cases} \frac{1}{\psi}\frac{\alpha_1}{\pi_1}B_{t-1} & \text{, with probability } \pi_1 \\ \frac{1}{\psi}\frac{\alpha_2}{\pi_2}B_{t-1} & \text{, with probability } \pi_2 \\ \frac{1}{\psi}\frac{1-\alpha_1-\alpha_2}{1-\pi_1-\pi_2}B_{t-1} & \text{, with probability } 1-\pi_1-\pi_2 \end{cases}, \tag{5}$$

where it is assumed that $0 < \alpha_1 < 1, 0 < \alpha_2 < 1, 0 < 1 - \alpha_1 - \alpha_2 < 1$, and additionally for the state probabilities, $(1-\alpha_1-\alpha_2)/(1-\pi_1-\pi_2) < \alpha_2/\pi_2 < \alpha_1/\pi_1$. The parameter restrictions imply that in States 1 and 2, we have $B_t > B_{t-1}$, whereas in State 3, we have $B_t < B_{t-1}$. Fukuta (1998) refers to State 1 as the "large bubble state", State 2 as the "small bubble state", and State 3 as the "incomplete burst state". The major empirical drawback of Fukuta's specification is that, within each of the three states, the bubble is subject to deterministic growth.

# 3 A new rational bubble

We now introduce a new rational bubble, which is strictly positive, recurringly explosive and stochastically deflating. With the same notation as for the Evans and the Fukuta bubbles from Eqs. (4) and (5), our specification consists of two distinct states:

$$B_t = \begin{cases} \frac{\alpha}{\psi\pi}B_{t-1}u_t & \text{, with probability } \pi \\ \frac{1-\alpha}{\psi(1-\pi)}B_{t-1}u_t & \text{, with probability } 1-\pi \end{cases}. \tag{6}$$

Via the Bernoulli process $\{\nu_t\}$, this mixture of distributions can be written in one single equation as

$$B_t = \left[\left(\left[\frac{\alpha}{\psi\pi} - \frac{1-\alpha}{\psi(1-\pi)}\right]\nu_t + \frac{1-\alpha}{\psi(1-\pi)}\right)B_{t-1}\right]u_t, \tag{7}$$

where we assume that $0 < \alpha < 1$. This latter constraint ensures that the bubble never collapses to zero and can thus reinflate.

Additionally, we stipulate $\frac{\alpha}{\pi} > 1$ and $\frac{1-\alpha}{1-\pi} < \psi$, ensuring the following neat interpretation of our two bubble states. In State 1, which occurs with probability $\pi$, the bubble grows with the mean factor $\frac{\alpha}{\psi\pi}$, implying the mean growth rate $\frac{\alpha}{\psi\pi}-1=\frac{\alpha}{\pi}-1+\frac{\alpha}{\pi}\cdot r>r$ (i.e. a faster growth rate than the required rate of return). State 2, occurring with probability $1-\pi$, models the deflation of the bubble with mean deflation factor $\frac{1-\alpha}{\psi(1-\pi)}<1$, or equivalently, with negative mean growth rate $\frac{1-\alpha}{\psi(1-\pi)}-1<0$. Hinging on the specific parameter constellation, the quantitative extent of the bubble deflation can range between a "small/moderate correction" and a "big crash" within one or arbitrarily many periods. In contrast to Fukuta's specification (5), our model allows for stochastic bubble growth/deflation within each state via the random variable $u_t$ in Eq. (6). In addition to its realistic trajectories, our specification is also more parsimonious than the Evans and Fukuta models.

It remains to prove the rationality of our bubble by verifying the martingale property (3). Using (i) the stochastic independence of the processes $\{u_t\}$ and $\{\nu_t\}$, (ii) the conditional unit-mean assumption for all $u_t$, and (iii) the Bernoulli distribution for all $\nu_t$, we readily obtain from the representation (7)

$$E_t(B_{t+1})=E_t\left\{\left[\left(\left[\frac{\alpha}{\psi\pi}-\frac{1-\alpha}{\psi(1-\pi)}\right]\nu_{t+1}+\frac{1-\alpha}{\psi(1-\pi)}\right)B_t\right]u_{t+1}\right\}=(1+r)\cdot B_t.$$

Figure 2 about here

Figure 2 displays four simulated trajectories of our stochastically deflating bubble process (6). In each simulation, we set $\psi=0.9804$ (which corresponds to a required rate of retrun of 2%), but choose distinct combinations of the parameters $\iota^2$, $\alpha$ and $\pi$. We initiate all bubble processes with the starting value $B_0=0.5$. The trajectories consist of 250 observations, representing a time span of approximately 21 years, based on monthly data. All trajectories exhibit two or three major bubbles, all differing from each other (i) in their respective stochastic growth rates during the build-up phases, and (ii) in their stochastic deflation rates during the downturns.

# 4 Bubbles and stock-price volatility

Rotermann and Wilfling (2014) analyze stock-price volatility in the presence of the Evans bubble (4). Assuming that the dividend payments in Eq. (2) follow a driftless random walk,

$$D_t = D_{t-1} + \varepsilon_t,$$

with $\{\varepsilon_t\}$ being an i.i.d. Gaussian white-noise process with mean zero and variance $\sigma^2$, the authors first show that the variance of the stock price $P_t$ from Eq. (2), conditional on all information available to market participants as of date $t-1$, is given by

$$\mathrm{Var}_{t-1}(P_t) = \mathrm{Var}_{t-1}(B_t) + \sigma^2/r^2. \tag{8}$$

Based on Eq. (8), the authors finally derive two main results on conditional stock-price volatility under the Evans bubble. (i) The Evans bubble triggers excess stock-price volatility. (ii) At the beginning of the bubbly period (i.e. before the Evans bubble becomes explosive), stock-price volatility is relatively low, whereas towards the end of the bubble and its bursting, stock-price volatility is typically high and reaches its maximum value one period after the crash.

While these two theoretical properties are consistent with empirical findings (Brunnermeier and Oehmke, 2013; Kindleberger and Aliber, 2005, pp. 24-37), conditional stock-price volatility dynamics under the Evans bubble exhibits a major drawback. The one-period collapse of the Evans bubble $\{B_t\}$ carries over to conditional stock-price volatility, which also collapses within one period after having attained its maximum value (Rotermann and Wilfling, 2014, Eq. (10)).

In order to assess conditional stock-price volatility under our new rational bubble, we use its representation (7) to obtain the conditional bubble variance as

$$\begin{aligned}
\mathrm{Var}_{t-1}(B_t) &= \mathrm{Var}_{t-1}\left[\left(\frac{\alpha}{\psi\pi} - \frac{1-\alpha}{\psi(1-\pi)}\right) B_{t-1}\nu_t u_t + \frac{1-\alpha}{\psi(1-\pi)} B_{t-1} u_t\right] \\
&= \left(\frac{\alpha B_{t-1}}{\psi\pi} - \frac{(1-\alpha)B_{t-1}}{\psi(1-\pi)}\right)^2 \cdot \mathrm{Var}_{t-1}(\nu_t u_t) \\
&\quad + \left(\frac{(1-\alpha)B_{t-1}}{\psi(1-\pi)}\right)^2 \cdot \mathrm{Var}_{t-1}(u_t) \\
&\quad + 2\left(\frac{\alpha}{\psi\pi} - \frac{1-\alpha}{\psi(1-\pi)}\right) \cdot \frac{1-\alpha}{\psi(1-\pi)} \cdot B_{t-1}^2 \cdot \mathrm{Cov}_{t-1}(u_t, \nu_t u_t). \qquad (9)
\end{aligned}$$

The distributional assumptions from Sections 2 and 3 enable us to find the variance-covariance terms in Eq. (9). Inserting Eq. (9) into Eq. (8), we obtain the conditional stock-price variance under our bubble specification (6):

$$\begin{aligned}
\mathrm{Var}_{t-1}(P_t) &= \left[\frac{(\alpha-\pi)^2}{\psi\pi(1-\pi)^2} \cdot \left(\exp\{\iota^2\} - \pi\right)\right. \\
&\quad \left. + \frac{(1-\alpha)^2 + 2(\alpha-\pi)(1-\alpha)}{\psi^2(1-\pi)^2} \cdot \left(\exp\{\iota^2\} - 1\right)\right] \cdot B_{t-1}^2 + \frac{\sigma^2}{r^2}. \qquad (10)
\end{aligned}$$

We note that the parameter restrictions stipulated in Section 3 imply $0 < \pi < \alpha < 1$, ensuring that the term in squared brackets in Eq. (10) is strictly positive. Thus, Eq. (10) constitutes a strictly monotone increasing relationship between the conditional stock-price variance $\mathrm{Var}_{t-1}(P_t)$ and $B_{t-1}$.

The monotone increasing relationship between $\mathrm{Var}_{t-1}(P_t)$ and $\{B_{t-1}\}$ causes a one-to-one propagation of the stochastic bubble deflation on conditional stock-price volatility dynamics. To illustrate this, we reconsider the NASDAQ index shown in Figure 1. Using monthly dividend data provided by *Datastream*, we estimated the parameters of our (unobservable) bubble specification (6) for the NASDAQ via a sequential Monte Carlo method—the particle filter—as introduced by Gordon et al. (1993). More specifically, to estimate our latent bubble process $\{B_t\}$, along with the parameters $\sigma^2, \psi, \iota^2, \pi$, and $\alpha$, we transformed the present-value formula (2), plus our bubble specification (6), into a nonlinear state-space representation, which we estimated using the *Expectation Maximization* (EM) algorithm as proposed by Schön et al. (2011).[2]

Table 1 about here

Table 1 displays the parameter estimates and standard errors, which we computed with the stable estimator of the information matrix, as established in Duan and Fulop (2011). We note that the parameter estimates satisfy the technical restrictions $\frac{\alpha}{\pi} > 1$ and $\frac{1-\alpha}{1-\pi} < \psi$, as imposed in Section 3.

Figure 3 about here

Figure 3 displays the estimated, stochastically deflating bubble trajectory $\{B_t\}$ (thin line) along with the associated conditional variance path $\mathrm{Var}_{t-1}(P_t)$ (bold line).[3] Three aspects concerning the conditional volatility trajectory, all stemming from Eq. (10), are worth mentioning. (i) The empirically consistent deflating behavior of our bubble (6) yields a conformable volatility trajectory, thus avoiding the one-period volatility collapse inherent in the Evans bubble. (ii) The stock-price variance $\mathrm{Var}_{t-1}(P_t)$ attains its (local) maximal values at that moment, when $B_{t-1}$ takes on its largest values. This means that stock-price volatility is typically maximal at the moment the bubble bursts (the beginning of the crisis). In Figure 3, we indicate the two bubble bursts by the vertical lines tagging the dates "March 2000" and "November 2007". (iii) Stock-price volatility starts deflating with a lag of one period after the bubble started deflating.

[2]Technical details of our estimation procedure are available upon request.
[3]We divided the NASDAQ time series by 10, in order to achieve numerical stability of the EM algorithm.

## 5 Conclusion

This paper establishes a new rational, recurringly explosive bubble that (i) is empirically plausible and parsimoniously parameterized, and (ii) merges all essential features of rational bubble specifications hitherto existing in the literature. In contrast to the benchmark Evans (1991) process, our bubble (i) restarts to grow from a variable base level after a crash, and (ii) generates stochastically deflating trajectories. In particular, these longer-lasting bubble deflation processes lead to stock-price volatility paths that are strongly consistent with real-world data. As an example, we use a particle-filter technique to extract the latent bubble component from NASDAQ data and analyze the volatility implications.

In the literature, the dominant approach to assessing bubbles consists of applying fixed-sample and sequential cointegration tests to a dividend stock-price relationship (see Phillips et al., 2015a, 2015b; and the literature cited there). These indirect bubble tests are designed to provide agents with binary yes/no information on whether the market is currently on an explosive bubble path. In this paper, we bring a complementary perspective to these mere time-series bubble tests, by estimating structural bubble specifications with real-world data. A useful line of future research could entail establishing a fully-fledged estimation framework with the objective of empirically assessing the dynamic properties of stock-market bubbles (such as mean bubble growth/deflation rates, bursting probabilities).

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## Tables and Figures

**Table 1**
Estimation results for the NASDAQ using the particle filter

| Parameter | Estimate | Standard error |
|---|---|---|
| $\sigma^2$ | 0.4476 | 0.0014 |
| $\psi$ | 0.9840 | $1.7480\times10^{-5}$ |
| $\iota^2$ | 0.0061 | $7.6905\times10^{-7}$ |
| $\pi$ | 0.9595 | 0.0011 |
| $\alpha$ | 0.9675 | $8.6242\times10^{-4}$ |
| $\frac{\alpha}{\pi}$ | 1.0083 | |
| $\frac{1-\alpha}{1-\pi}$ | 0.8024 | |

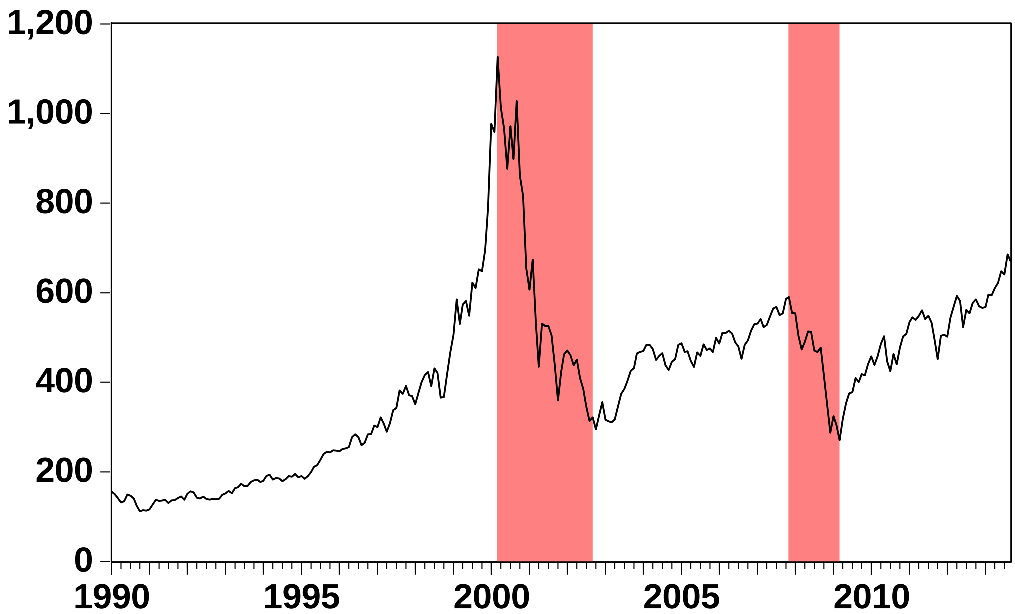


**Figure 1. NASDAQ stock-market index, January 1990 – October 2013**

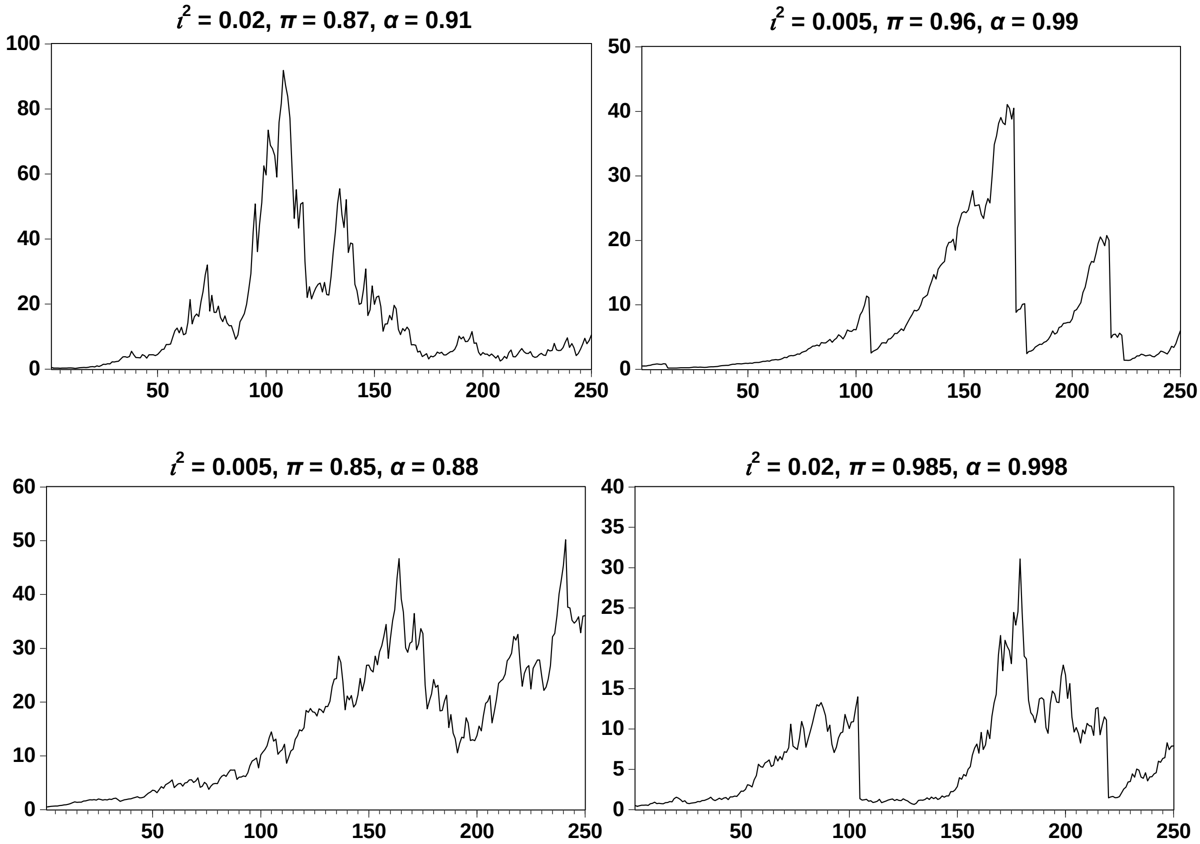


**Figure 2. Bubble trajectories simulated according to Eq. (6)**

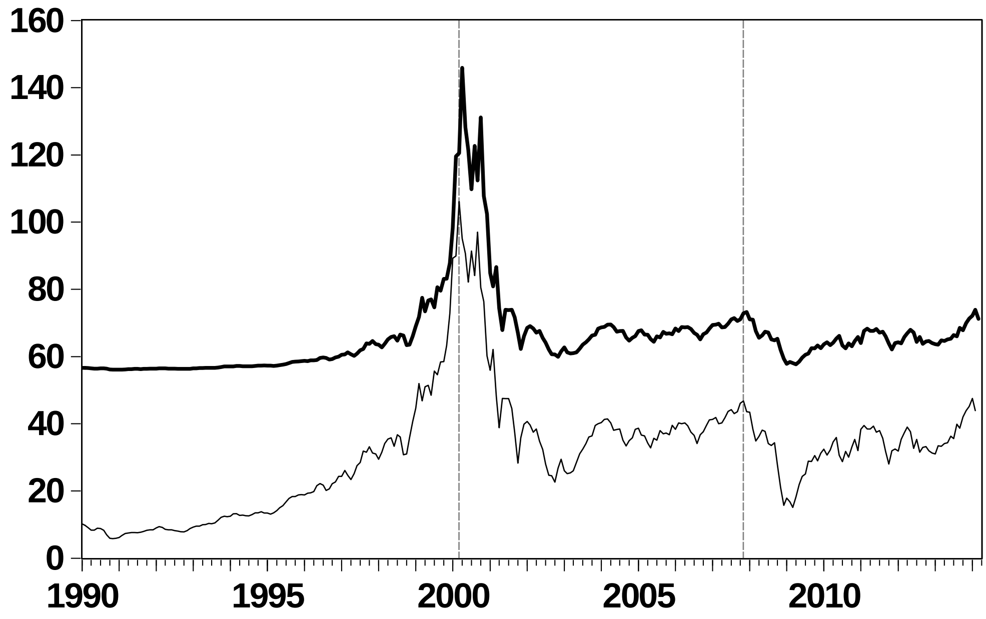


**Figure 3. NASDAQ, estimated bubble trajectory (thin line) and conditional variances (bold line) according to Eq. (10)**