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Abstract

The Gain-Loss-Ratio, proposed by Bernardo and Ledoit (2000), evaluates the attractiveness of an investment opportunity for an investor with a given stochastic discount factor. It can either be used as a performance measure on a market with known prices or to derive price-intervals in incomplete markets. For both applications, there is a considerable theoretical drawback: It reaches infinity for nontrivial cases in many standard models with continuous probability space.
In this paper, a more general ratio is proposed, which includes the original Gain-Loss-Ratio as a limit case. This so-called "Substantial Gain-Loss-Ratio" is applicable in case of continuous probabilities. In addition, in its function as a performance measure it helps illuminate the source respectively the distribution of out-performance, that a portfolio reveals.

JEL classification: G11, G12, G13
Keywords: Gain-loss ratio, acceptability index, incomplete markets, good-deal bounds

I. Introduction

The Gain-Loss-Ratio, proposed by Bernardo and Ledoit (2000), evaluates the attractiveness of an investment opportunity for an investor with a given stochastic discount factor (SDF hereafter). Originally, it was derived to fill the gap between model-based pricing and no-arbitrage pricing[1] and can thus be classified as part of the "Good-Deal-Bound" literature. This literature derives price-bounds by precluding SDFs which would create too attractive assets, in the sense of their Good-Deal measure, and thereby inclining the no-arbitrage price bounds. Cochrane and Saa-Requejo (1998) introduced the terminology "Good-Deal" in this context and were the first who derived Good-Deal-price bounds for option-prices by restricting the Sharpe ratio. Cherny (2003) generalized their approach and made it well-applicable for skewed assets.
Since arbitrage opportunities do not imply large Sharpe ratios,[2] one needs to check for two conditions to get narrowed no-arbitrage and no-good-deal bounds. Regarding this, the Gain-Loss-Ratio (GLR hereafter) approach provides several significant advantages. A bounded GLR implies the absence of arbitrage opportunities. Another advantage, which might be the main reason for the success of the GLR approach, results from its asset-pricing model foundation. The GLR quantifies the attractiveness of an investment concerning a benchmark investor, e.g. a representative investor of a specified asset-pricing model.
Alongside the determination of price bounds on incomplete markets, the GLR can be used

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[1] Cp. Bernardo and Ledoit (2000) p. 144.

[2] Cp. e.g. Dybvig and Ingersoll (1982) who show that in the CAPM, with its bounded Sharpe ratios, arbitrage opportunities do exist.

to compare asset-pricing models[3] or to measure the performance of funds in an economically meaningful way, i. e. concerning a benchmark SDF. In addition to the stated theoretical foundation and the preclusion of arbitrage opportunities, the GLR has many other desirable properties. It is not only very intuitive but it also fulfills a lot of requirements of a good performance measure, e. g. all conditions of an "acceptability index" in the sense of Cherny and Madan (2009).
Nevertheless, the GLR approach has one notable drawback: In many standard models, e. g. the Black-Scholes model, the best GLR becomes infinity. One might argue that the GLR is only constructed for discrete probability space, but even then the strong dependence on the specific discretization can be seen as a severe drawback.
The goal of this paper is to overcome the aforementioned drawbacks and to thereby strengthen the concept by introducing the so-called *Substantial-Gain-Loss-Ratio* (SGLR hereafter). The SGLR is a tool that allows working in continuous probability spaces without loosing the positive properties of the GLR. The determination of price-intervals as well as the comparison of a portfolio performance in the light of different asset-pricing models on a theoretical level in models with continuous probability spaces becomes possible. This is achieved by slightly changing the condition for a Good-Deal, i. e. on the most extreme but at the same time very small part of the statespace.
Additionally, the source or distribution of differences in performance is illuminated via so-called *β-diagrams*.
The paper proceeds as follows: Section two introduces the SGLR and the β-diagram. Section three discusses the properties as a performance measure. Section four explains the application for price bounds in incomplete markets and section five concludes.

II. The Substantial-Gain-Loss-Ratio and the β-diagram

In an asset pricing context, the price $p(X)$ of an asset with a random payoff X tomorrow is usually stated as $p(X) = E[MX]$ with a SDF M. E. g. consumption-based asset-pricing models interpret the SDF as the ratio of marginal utility of consumption tomorrow to marginal utility of consumption today.[4] The intuition is that investments are more valuable if they have relatively large payoffs in states of low consumption than those that mainly pay in "good" states.[5]
In this setting, given a continuous state space, it is assumed that the full distribution of future states and its prices in form of the SDF is known and that any arbitrary unlikely event can influence the price of an investment, via a sufficient large corresponding SDF, significantly. The SGLR overcomes this problem by taking the infimum over all GLRs with positive benchmark SDFs M' which are substantially equal to to the original benchmark SDF M^*. Substantially equal means that they do only differ in a small number of states, i.e. $P(M' \neq M^*) \leq \beta$ for a small $\beta > 0$ and the deviation is close to the original one, i.e. $Var(M') \leq Var(M^*) + \beta$.[6] The definition is given by:

Definition II.1. *Given an ordinary market-model $((X_i)_{i\in I}, (\Omega, \mathcal{F}, P), p)$ with investment opportunities with payouts $(X_i)_{i\in I}$, probability space $(\Omega, \mathcal{F}, P)$ and pricing function p.*[7]
Let $SDF^+_\beta(M^*) := \{M'$ is positive SDF $: (P(M' = M^*) \geq 1-\beta) \wedge (Var(M') \leq Var(M^*) + \beta)\}$ and $\chi := \{X_i : i \in I \ \wedge X_i \not\equiv 0\}$. The following coefficient $SGLR^{M^*}_\beta((X_i)_{i\in I})$ is called Substantial-Gain-Loss-Ratio of the investment opportunities I to the subsistence level $1-\beta$ with respect to the benchmark SDF

[3] In the context of model risk, i.e. the compared models may not price all assets correctly, the best GLR can be used as a measure of misspecification.
[4] Cp. Cochrane (2001) chapter 1.1.
[5] Given an increasing and concave utility function.
[6] In appendix B a parsimonious example for a slightly changed SDF, its impact on the GLR and a graphical illustration are given.
[7] The ordinary market-model is defined in appendix A definition A.1 and is introduced for a better readability.

M^*:[8]

$$SGLR_\beta^{M^*}((X_i)_{i\in I}) := \sup_{X_i\in\mathcal{X}} \inf_{M'\in SDF_\beta^+(M^*)} \frac{E[(M'(X_i - p(X_i)))^+]}{E[(M'(X_i - p(X_i)))^-]} \tag{1}$$

For single investments with payout X_i the following notation is provided:

$$SGLR_\beta^{M^*}(X_i) := SGLR_\beta^{M^*}(\{aX_i | a \in \mathbb{R} \setminus \{0\}\})$$

Including the inverse investment means for the SGLR of a single investment that the attractiveness of the more attractive position (short or long) is evaluated.
The "normal" GLR of a zero-cost investment with payout X_i is given by $SGLR_0^{M^*}(\{X_i\})$ with respect to the SDF $M^* \equiv 1$ and the "best GLR" of a market I is given by $SGLR_0^{M^*}((X_i)_{i\in I})$ with respect to the same SDF.
If a SDF M^* prices all assets of I correctly, the $SGLR$ is smaller or equal one. If the β-SGLR is one, it means that M^* prices all considered assets substantially correctly, i.e. by slightly[9] changing M^* in up to β of the states. A $SGLR$ larger than one means that M^* does not price the assets of the market correctly. Hence the $SGLR_\beta^{M^*}((X_i)_{i\in I})$ quantifies the substantial mispricing of the whole market concerning the benchmark SDF. In the special case of $SGLR_\beta^{M^*}(X_i)$, i.e. when focusing on one asset only, the SGLR quantifies the attractiveness of the asset[10] for an investor with corresponding SDF, possibly evaluating the payout distribution on a part of the states, with up to β of the probability mass, with a changed SDF.
One motivation for this definition is that market participants are not able to see/assess the entire distribution of future states, e.g. events that only happen every 1000 years are not observable for the investors and therefore do not have an impact on the price. Put differently, the price is only influenced by substantial experiences/scenarios.
Critics who do not accept this permission of "nescience" or "freedom" should not comprehend the SGLR as a tool of ignorance, but rather as a tool that illuminates the impact of these unlikely events on the attractiveness of assets.
Changing β allows to explicitly investigate the probability disbalance of Gains and Losses. This dependence of the SGLR can be displayed with a so-called β-diagram.

Definition II.2. *Given the situation of definition II.1, a plot of $SGLR_\beta^{M^*}((X_i)_{i\in I})$ against β is called β-diagram.*

Figure 1 shows the β-diagram of the portfolio $Prtf_A$ given in example II of appendix C. This diagram shows that large parts of the attractiveness are due to the payouts in very rare events. Smoothing the impact of the rare events, i. e. increasing beta, strongly reduces the $SGLR$.
This section shall close with a theorem that states the finiteness of the SGLR.

Theorem II.1. *Given the situation of definition II.1, let $M^* > 0$ be the benchmark SDF, let $\beta > 0$ and let $\mathcal{M}_p$ be the set of all SDF that prices the assets correctly.*
If I is arbitrage-free and $\frac{\int_{A_n} m^2 \, dP}{(\int_{A_n} m \, dP)^2}(\int_{A_n} M^* \, dP)^2 \overset{n\to\infty}{\to} 0$ *, with* $A_n := \{\frac{m}{M^*} \leq \frac{1}{n}\}$*, then the* $SGLR_\beta^{M^*}((X_i)_{i\in I})$

[8] In cases where there is no confusion about I or M^*, it is abbreviated β-SGLR. In this paper interest rates are not considered. The subtraction of the price just replaces the assumption that the asset has the price zero assumed w.l.o.g. for the GLR. For taking interest rates into account, the price must be multiplied by the interest rate.

[9] Slightly means here that the variance is not larger than the original one plus β.

[10] More exactly the attractiveness of a short or long position, depending on which is more attractive, is quantified.

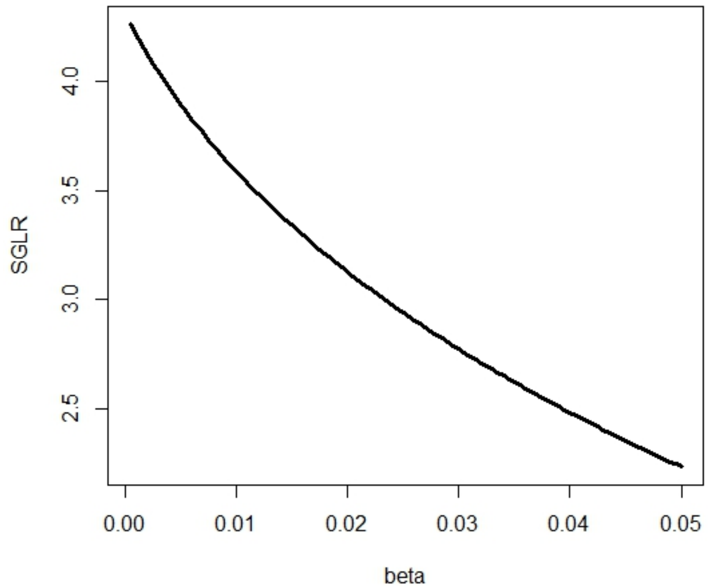

Figure 1: *β-diagram of the portfolio $Prtf_A$ given in example II of appendix C*

is bounded:

$$SGLR_\beta^{M^*}((X_i)_{i\in I}) \leq \inf_{m\in\mathcal{M}} c(m) < \infty,$$

with an appropriate function $c : \mathcal{M} \longrightarrow \mathbb{R}^+$.

Proof. The following notation is introduced:
Let SDF^+ be the set of all positive SDFs, $M_1, M_2 \in SDF^+$ and $\beta^* > 0$

$$\overline{C}(M_1, M_2) := \{\omega \in \Omega : \frac{M_1}{M_2} \geq (1 - \frac{\beta^*}{2}) - \text{quantile of } \frac{M_1}{M_2}\}$$
$$\underline{C}(M_1, M_2) := \{\omega \in \Omega : \frac{M_1}{M_2} \leq \frac{\beta^*}{2} - \text{quantile of } \frac{M_1}{M_2}\}$$

[11] further

$$\overline{c}(m, M^*) := \frac{\int_{\overline{C}(m,M^*)} M^* dP}{\int_{\overline{C}(m,M^*)} m dP}$$
$$\underline{c}(m, M^*) := \frac{\int_{\underline{C}(m,M^*)} M^* dP}{\int_{\underline{C}(m,M^*)} m dP}$$

[11] As we have a continuous statespace $P(\Omega \setminus (\overline{C} \bigcap \underline{C})) = 1 - \beta^*$ holds.

and eventually

$$\underline{\overline{M'}}(m) := \mathbb{1}_{\Omega\setminus\underline{C}(m,M^*)\cap\overline{C}(m,M^*)}M^* + \mathbb{1}_{\underline{C}(m,M^*)}m\underline{c} + \mathbb{1}_{\overline{C}(m,M^*)}m\overline{c}.$$

hence $\underline{\overline{M'}} \in SDF^+_\beta(M^*)$ for a sufficiently small β^*, which exists according to Lemma A.1.
W.l.o.g. we assume that $p(X_i) = 0$ f.a. $X_i \in \chi$.[12]
Then the following is true:

$$\begin{aligned}
SGLR^{M^*}_\beta((X_i)_{i\in I}) &= \sup_{X_i\in\chi} \inf_{M'\in SDF^+_\beta(M^*)} GLR^{M'}(X_i) \\
&\leq \sup_{X_i\in\chi} GLR^{\underline{\overline{M'}}(m')}(X_i) \text{ f.a. } m' \in \mathcal{M}_p \\
&\overset{\star}{\leq} \inf_{m\in\mathcal{M}_p} \frac{ess\sup \frac{m}{\underline{\overline{M'}}(m)}}{ess\inf \frac{m}{\underline{\overline{M'}}(m)}} \\
&= \inf_{m\in\mathcal{M}_p} \frac{max(ess\sup_{\Omega\setminus\overline{C}} \frac{m}{\underline{\overline{M'}}(m)}, ess\sup_{\overline{C}} \frac{m}{\underline{\overline{M'}}})}{min(ess\inf_{\Omega\setminus\underline{C}} \frac{m}{\underline{\overline{M'}}(m)}, ess\inf_{\underline{C}} \frac{m}{\underline{\overline{M'}}})} \\
&= \inf_{m\in\mathcal{M}_p} \frac{max(ess\sup_{\Omega\setminus\overline{C}} \frac{m}{\underline{\overline{M'}}(m)}, \frac{m}{m\overline{c}(m))}}{min(ess\inf_{\Omega\setminus\underline{C}} \frac{m}{\underline{\overline{M'}}(m)}, \frac{m}{m\underline{c}(m)})} \\
&= \inf_{m\in\mathcal{M}_p} \frac{max(ess\sup_{\Omega\setminus\overline{C}} \frac{m}{\underline{\overline{M'}}(m)}, \frac{1}{\overline{c}(m)})}{min(ess\inf_{\Omega\setminus\underline{C}} \frac{m}{\underline{\overline{M'}}(m)}, \frac{1}{\underline{c}(m)})} \\
&= \inf_{m\in\mathcal{M}_p} \underbrace{\frac{\underline{c}(m)}{\overline{c}(m)}}_{=:c(m)} < \infty
\end{aligned}$$

where $\star$ is true because of the duality theorem for GLR.[13] □

[12]This is allowed since any asset i with payout X_i and price $p(X_i)$ has the same SGLR as an asset with price zero and payout $X_i - p(X_i)$.
[13]Cp. Biagini and Pinar (2013) section 2.2.

III. The SGLR as a performance measure

There are performance measures with different properties that might be suitable for different occasions.[14] A new performance measure like the SGLR must therefore justify its purpose. It does so in a twofold way. Firstly, it improves an already accepted performance measure, namely the GLR by Bernardo and Ledoit (2000) and secondly, it has many properties, which are desirable for a performance measure.
The GLR in its simplest form meets all eight criteria that a good performance measure should have according to Cherny and Madan (2009),[15] but it has the essential drawback that in many continuous standard models the best GLR of the market is infinity. Biagini and Pinar (2013) discuss the "pros and cons" of the GLR in detail and argue that the best GLR is a poor performance measure. Here we want to expose why the GLR might not be suitable as a performance measure in many occasions by considering some examples.
The first example is taken from Biagini and Pinar (2013) chapter 2.4. Given is a Black-Scholes market model, i. e. the asset Price S_T at time T is given by $S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$, where W_t is a standard Wiener process and $\mathcal{F}_t$ the corresponding Wiener filtration. Assuming w.l.o.g. that the interest rate is zero, the market price of risk π is given by $\pi = \frac{\mu}{\sigma}$ and the unique SDF for T is given by:

$$M_T = e^{-\pi W_T - \frac{\pi^2 T}{2}} \tag{2}$$

For $\mu \neq 0$ the market of all assets with payoffs in $\chi = \{X \in \mathcal{L}^2 : X \text{ is } \mathcal{F}_T \text{ measurable }\}$ and with the price function p given by $p(X) := E[M_T X]$ is arbitrage-free, but arbitrarily high GLRs will be attained in this market. We want to take a closer look on the GLR of digital put options. Since $M_T > K \iff S_T < f(K)$ for all $K \in \mathbb{R}$ and a proper function f, an asset with payoff $\mathbb{1}_{\{M_T > K\}} - \underbrace{E[M_T \mathbb{1}_{\{M_T > K\}}]}_{p_K}$ can be seen as a digital option minus price on the asset with strike $f(K)$. The short-position of this option has the payout $Z_k = p_k - \mathbb{1}_{\{M_T > K\}}$ and since $1 > p_K > P(M_T > K)K$ the GLR of it is given by:[16]

$$\frac{E[Z_K^+]}{E[Z_K^-]} = \frac{p_K(1 - P_K)}{(1 - p_K)P_k} > K(1 - P_K),$$

where $P_K := P(M_T > K)$. Hence already in this simple model the GLRs are unbounded. Therefore it is easy to manipulate the GLR as a performance measure, e. g. adding a digital put with extremely low strike will increase the GLR of a portfolio.
This result can be easily understood by the duality formulation of the best GLR in a market with payoffs χ. Under some assumptions[17] and the case of a unique "true" SDF M and benchmark SDF M^* the following holds

$$\sup_{X \in \chi} GLR(X) = \frac{ess\ sup \frac{dQ}{dQ^*}}{ess\ inf \frac{dQ}{dQ^*}},$$

[14]For a survey see Caporin et al. (2013).
[15]The properties are discussed later for the SGLR and can be looked up including the proofs for GLR in chapter 3.2. in the paper of Cherny and Madan (2009). With "the simplest form of the GLR" $\frac{E[X]}{E[X^-]}$ is meant.
[16]The benchmark SDF here is just constantly one.
[17]See Biagini and Pinar (2013) p. 6 .

with $dQ = MdP$ and $dQ^* = M^*dP$.[18] It means that the best GLR is determined by the behavior of the random variable $\frac{M}{M^*}$ in the neighborhood of zero and its right tail; i.e. by the states in which the highest deviations between the price-giving SDF and the benchmark SDF occur. This shows that whenever $Q \not\gg Q^*$ or the support of $\frac{Q}{Q^*}$ is unbounded the best GLR is infinity.
A discretization of the model leads to a finite best GLR, but the result is highly dependent on the kind of discretization. In practical computation of the GLR, the theoretical state space Ω will be discretized to Ω_D and the dual formulation mentioned before becomes[19]

$$\frac{\max\limits_{\omega \in \Omega_d} \frac{M_d(\omega)}{M_d^*(\omega)}}{\min\limits_{\omega \in \Omega_d} \frac{M_d(\omega)}{M_d^*(\omega)}}.$$

While usually just a question of computational correctness, here the choice of discretization determines the bounds of the best GLR and allows to manipulate the measurement of performance in an extensive way. For portfolios with a distribution with asymmetric fat tails, unlikely but large losses can be "discretized away" by cutting high losses to a maximal level. Discretization of that kind reduces the expected discounted loss and hence the GLR increases. Respectively unlikely but high potential returns can be emphasized by a suitable discretization. Moderate risk portfolios with e.g. a normal distribution might be made more attractive in comparison by aggregating unlikely losses of the fat-tailed distributed portfolios in an exaggerated way. This will happen implicitly in the computation and can not be seen directly when comparing GLRs of different portfolios or funds. Of course similar is true for the period of observation and the method of estimation, which might treat unlikely events in a different manner.
Hence implicitly, any practical application of the GLR does ignore parts of a potential continuous distribution. This has especially serious consequences when distributions are not symmetric, e.g. in the case when options or other derivatives are included.
Both problems, i. e. the arbitrarily high attractiveness in a theoretical framework on the one hand, and the possibility of manipulations - respectively the implicit assumption in a discretized framework - on the other hand, can be solved by the SGLR. By explicitly changing the impact of some part of the state space, manipulations become visible, theoretical continuous models can be analyzed as the SGLR is not infinity and differences in the dependence of the performance on small but extreme parts can be investigated through the $\beta-$diagram.
In appendix C two different portfolios are given, which β-diagrams can be seen in Figure 2. The graphs show that it is possible that the GLR of portfolio $Prtf_B$ is larger than the one of portfolio $Prtf_A$, but there exist investors, which only differ from the original benchmark investor slightly, for which $Prtf_A$ is more attractive.[20] Hence the performance advantage of $Prtf_B$ is due to a very small and extreme part of the distribution and it may take a long time until this advantage will realize. The attractiveness of a portfolio in a SGLR context is dependent on β like the performance in a GLR context is dependent on the precision of the discretization. The β-diagram makes these dependencies transparent.
In appendix C an additional example is given that shows that a portfolio can have a finite price and an undefined GLR, but a finite β-SGLR for $\beta > 0$.
After seeing the drawbacks of the GLR and its improvement by the SGLR, we want to show a number of properties of the SGLR as a performance measure. Therefore we look at the criteria which, as argued by Cherny and Madan (2009) a measure of trading performance should satisfy

[18] For the risk-neutral benchmark SDF $dQ^* = 1dP = dP$ holds.
[19] M_d denotes the discretized SDF.
[20] E.g. there are investors with SDF $\in SDF_{0.02}(M^*)$, such that $\frac{E[M'Prtf_A]^+}{E[M'Prtf_A]^-} > \frac{E[M'Prtf_B]^+}{E[M'Prtf_B]^-}$ holds.

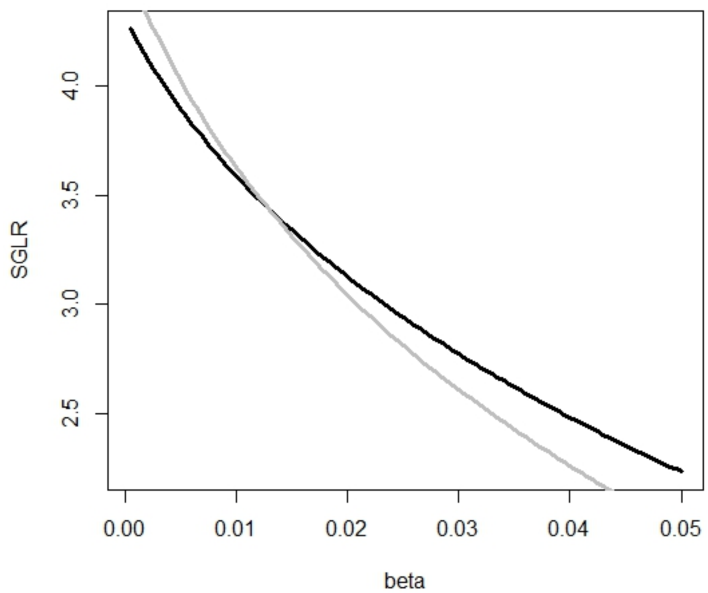

Figure 2: *β-diagrams of portfolio $Prtf_A$ and portfolio $Prtf_B$ given in example III in appendix C*

in order to highlight the properties of the SGLR.
Cherny and Madan (2009) provide eight properties which are examined in the next eight propositions. For all propositions let the situation of definition II.1 be given, and let $X_i, X_j, (X_n)_{n\in\mathbb{N}}$ be nonzero payouts of the stochastic investments of the market I, where w.l.o.g. $p(X_i) = p(X_j) = p(X_n) = 0$ is true.

Proposition III.1. *(Quasi-concavity):*

$$(SGLR^{M^*}_\beta(X_i) \geq a \text{ and } SGLR^{M^*}_\beta(X_j) \geq a) \Rightarrow \quad SGLR^{M^*}_\beta(\lambda X_i + (1-\lambda)X_j) \geq a \text{ for any } \lambda \in [0,1] \tag{3}$$

Proof. Let $SGLR^{M^*}_\beta(X_i) \geq a$ and $SGLR^{M^*}_\beta(X_j) \geq a$, then the following holds for all $M' \in SDF^+_\beta(M^*)$:

$$E[(M'X_i)^+] \geq aE[(M'X_i)^-]$$
$$E[(M'X_j)^+] \geq aE[(M'X_j)^-]$$

hence:

$$E[M'X_i] \geq 2aE[(M'X_i)^-] \tag{4}$$
$$E[M'X_j] \geq 2aE[(M'X_j)^-] \tag{5}$$

due to the convexity of $(\cdot)^-$ it further holds:

$$\begin{aligned}
2aE[(\lambda M' X_i + (1-\lambda)X_j M')^-] &\le 2aE[(\lambda X_i M')^- + ((1-\lambda)X_j M')^-] \\
&= 2a\lambda E[(X_i M')^-] + (1-\lambda)E[(X_j M')^-] \\
&\overset{(4),(5)}{\le} \lambda E[(X_i M')] + (1-\lambda)E[X_j M'] \\
&= E[(\lambda X_i M') + (1-\lambda)X_j M']
\end{aligned}$$

$$\begin{aligned}
\iff aE[(\lambda X_i M' + (1-\lambda)X_j M')^-] &\le E[(\lambda X_i M' + (1-\lambda)X_j M')^+] \\
\iff \frac{E[(\lambda X_i M' + (1-\lambda)X_j M')^+]}{E[(\lambda X_i M' + (1-\lambda)X_j M')^-]} &\ge a,
\end{aligned}$$

for all $M' \in SDF^+_\beta(M^*)$. □

Proposition III.2. *(Monotonicity):*

$$X_i \le X_j \textit{ almost surely } \implies SGLR^{M^*}_\beta(X_i) \le SGLR^{M^*}_\beta(X_j)$$

Proof.

$$\begin{aligned}
X_i \le X_j \text{ a.s.} \implies \quad & E[(M'X_i)^+] \le E[(M'X_j)^+] \\
& \text{and } E[(M'X_i)^-] \ge E[(M'X_j)^-] \text{ f. a. } M' \in SDF^+_\beta(M^*) \\
\implies \quad & SGLR^{M^*}_\beta(X_i) \le SGLR^{M^*}_\beta(X_j)
\end{aligned}$$

□

Proposition III.3. *(Scale invariance):*

$$SGLR^{M^*}_\beta(\lambda X_i) = SGLR^{M^*}_\beta(X_i) \textit{ f.a. } \lambda > 0 \tag{6}$$

Proof. Given $\lambda > 0$ the following holds:[21]

$$\begin{aligned}
SGLR^{M^*}_\beta(\lambda X_i) &= \inf_{M' \in SDF^+_\beta(M^*)} \frac{E[(M'\lambda X_i)^+]}{E[(M'\lambda X_i)^-]} \\
&= \inf_{M' \in SDF^+_\beta(M^*)} \frac{\lambda E[(M'X_i)^+]}{\lambda E[(M'X_i)^-]} \\
&= \inf_{M' \in SDF^+_\beta(M^*)} \frac{E[(M'X_i)^+]}{E[(M'X_i)^-]} \\
&= SGLR^{M^*}_\beta(X_i)
\end{aligned}$$

□

[21] Remember that w.l.o.g. $p(X_i) = 0$ holds.

Proposition III.4. *(Fatou property):*

$$(\exists a > 0 : |X_n| \leq 1 \text{ and } SGLR_\beta^{M^*}(X_n) \geq a \text{ for all } n \in \mathbb{N} \text{ with } X_n \xrightarrow{P} X_i) \Longrightarrow SGLR_\beta^{M^*}(X_i) \geq a \quad (7)$$

Proof. Given $\exists a > 0 : |X_n| \leq 1$ and $SGLR_\beta^{M^*}(X_n) \geq a$ for all $n \in \mathbb{N}$ with $X_n \xrightarrow{P} X_i$ the following holds:

$$\begin{aligned}
|X_n| \leq 1 \text{ and } E^{M'}[1] = 1 < \infty \text{ f.a. } M' \in SDF_\beta^+(M^*) \\
\Longrightarrow (X_n)_{n\in\mathbb{N}} \text{ uniformly integrable with respect to } P^{M'} \\
\Longrightarrow E^{M'}[(X_n)^\pm] \xrightarrow{P} E^{M'}[(X_i \mathbb{1}_B)^\pm],
\end{aligned}$$

where $P^{M'}$ and $E^{M'}$ are the probability and the expectation induced by the SDF M'.
$\frac{E[(M'X_n)^+]}{E[(M'X_n)^-]}$ is bounded and therefore:

$$\begin{aligned}
SGLR_\beta^{M^*}(X_i) &= \inf_{M' \in SDF_\beta^+(M^*)} \frac{E[(M'X_i)^+]}{E[(M'X_i)^-]} \\
&= \inf_{M' \in SDF_\beta^+(M^*)} \lim_{n\to\infty} \frac{E[(M'X_n)^+]}{E[(M'X_n)^-]} \\
&= \lim_{n\to\infty} \inf_{M' \in SDF_\beta^+(M^*)} \frac{E[(M'X_n)^+]}{E[(M'X_n)^-]} \\
&= \lim_{n\to\infty} SGLR_\beta^{M^*}(X_n) \geq a
\end{aligned}$$

□

Performance measures which fulfill the last four properties are called acceptability indexes. Hence the SGLR is an acceptability index. Cherny and Madan (2009) state four additional properties that they deem necessary for a good performance measure. The SGLR does not meet all of them. Nonetheless they are discussed in the following. Three of the requirements will be attacked, whereas the last holds.

Proposition III.5. *(Law variance):*
*The SGLR is **not** law invariant. I. e.:*

$$X_i \stackrel{law}{=} X_j \not\Longrightarrow SGLR_\beta^{M^*}(X_i) = SGLR_\beta^{M^*}(X_j) \quad (8)$$

Proof. The SGLR of X_i resp. X_j does depend on the covariance with M^* which is not determined by the single distribution of X_i resp. X_j. □

Cherny and Madan (2009) plead for law invariance, but it is the author's opinion that the dependence of the performance measure on the covariance of the payouts with the state, e.g. level of consumption via the SDF is a property that is more desirable. In his opinion, a portfolio that has high payouts in states of shortage and low payouts in times of prosper should be declared as better performing than a portfolio with the same distribution but high and low pay outs vice versa.

Proposition III.6. *(Inconsistency with second-order stochastic dominance):*
*The SGLR is **not** consistent with second-order stochastic dominance. I. e.:*

$$E[U(X_i)] \leq E[U(X_j)] \text{ for any increasing concave function } U \not\Longrightarrow SGLR_\beta^{M^*}(X_i) \leq SGLR_\beta^{M^*}(X_j) \quad (9)$$

Proof. Direct implication of the law variance. □

The idea of consistency with second-order stochastic dominance is to make the performance measure consistent with the Expected-Utility-Theory. The violation here is due to the even stronger bounds to Expected-Utility-Theory that consider the state, e.g. the consumption level which determines the total utility. So again it is the author's opinion that the violation of this property is due to a property that is more desirable than the required one.

Proposition III.7. *(Expectation Inconsistency):*
*The SGLR is **not** expectation consistent. I. e.:*

$$E[X_i] < 0 \not\Longrightarrow SGLR_\beta^{M^*}(X_i) = 0 \quad (10)$$

Proof. Since the SGLR is larger than zero for assets with $P(X_i > 0) > \beta$, the proposition follows. □

In the author's opinion, an asset can be attractive even if it has a negative expectation, namely if it has high payouts in time of shortage and stress. Hence he does not regard this property as a lack of quality but as a result of a property (state consideration) that is more desirable.

Proposition III.8. *(Arbitrage Consistency):*
The SGLR is arbitrage consistent. I. e.:

$$P(X_i \geq 0) = 1 \Longleftrightarrow SGLR_\beta^{M^*}(X_i) = \infty \quad (11)$$

Proof. In the case of $P(X_i \geq 0) = 1$ the denominator of $\frac{E[(M'X_i)^+]}{E[(M'X_i)^-]}$ becomes zero for any $M' \in SDF^+$ and therefore $SGLR_\beta^{M^*}$ infinity.
Vice versa, an infinite SGLR means that its denominator must be zero, since choosing a bounded M' ensures that the numerator is finite.[22] $E[(M'X_i)^-] = 0$ implies $P(X_i \geq 0)$, which means that X_i is an arbitrage opportunity. □

This last property is very important for using of the SGLR in the context of Good-Deal-Bounds. It ensures that a Good-Deal-free market, i.e. a market where all assets have a SGLR smaller than a certain bound, are arbitrage-free as well. In the next section the calculation of price intervals based on this idea is described.

[22]For $\beta > 0$ and $M^* \in SDF^+$ there is always a bounded $M' \in SDF_\beta^+(M^*)$ (cp. $\overline{M^*}$ from the proof of theorem II.1).

IV. Finding Price Intervals with the SGLR

The derivation of price bounds based on the SGLR is straightforward and analogous to the procedure described by Bernardo and Ledoit (2000) in section five. In the situation of definition II.1, price bounds can be calculated in the following way:[23]

1. The $SGLR_{\beta}^{M^*}((X_i)_{i\in I})$ of the market I is determined (, e. g. estimated).

2. The price interval for an asset with payout z is given by:

$$\sup_{u\in\mathbb{R}^+:\forall i\in I:SGLR_{\beta}^{M^*}(z-X_i)\geq SGLR_{\beta}^{M^*}((X_i)_{i\in I})\text{ with }p(z)=u} u \leq p \leq \inf_{v\in\mathbb{R}^+:\forall i\in I:SGLR_{\beta}^{M^*}(X_i-z)\geq SGLR_{\beta}^{M^*}((X_i)_{i\in I})\text{ with }p(z)=v} v$$

In appendix II the R-code to a practical example is given.[24] This example should be briefly explained here. In favor of generality, we will focus on a Monte-Carlo approach. In cases where analytical solutions can be derived, they should of course be preferred. We assume that the situation of definition II.1 is given and I consists of a stock and a call option on the stock. Trades can only be done at $t = 0$ and $t = T$. No intermediate trading is allowed. We assume that the stock price can be observed and its dynamic is given by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t + (-\gamma_j \mu_j dt + (e^{J_t} - 1)dN_t)S_t$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dB_t$$

where B_t and W_t are possibly correlated Wiener processes and N_t is a poisson-process with intensity γ_J and a lognormally distributed jump size $(e^{J_t} - 1)$. This means it belongs to the affine class by definition of Duffie et al. (2003). Price intervals for a call option can be derived in the following way:

1. A high number N of stock-price trajectories are simulated and S_T^i is saved for each simulation $i \leq N$.

2. A bound b for the SGLR is fixed.

3. The price of a call option $p(C)$ with strike K is then bounded in the following way:[25]

$$\max_{\omega_0\in[-1,1]:\widehat{SGLR}_{\beta}((\omega_0 * (S_T - S_0) + (1-\omega_0)((S_T-K)^- - x)))\geq b} x \leq p(C)$$
$$p(C) \leq \min_{\omega_0\in[-1,1]:\widehat{SGLR}_{\beta}((\omega_0 * (S_T - S_0) - (1-\omega_0)((S_T-K)^- - y))\geq b} y,$$

   where $\widehat{SGLR}_{\beta}((X_i)_{i\leq N})$ is an estimator for the SGLR of a single asset with payout X.

Instead of the simulated data historical data can be used. A general estimation of the $SGLR$ is not discussed in this paper as it is highly dependent on the benchmark SDF.[26]

[23] Under the assumption that the SGLR does not increase when the asset is added to the market.

[24] The example is geared to the example given in Bernardo and Ledoit (2000) chapter five. It is extended by stochastic volatility and jumps to underline its uncomplicated implementation even in cases of complex dynamics.

[25] $\omega_0 \in [-1, 1]$ holds w.l.o.g. since the SGLR is scale invariant.

[26] In the case of a risk-neutral benchmark investor an estimation algorithm is given in appendix D.I.

V. Conclusion

This paper introduces new tools, namely the Substantial-Gain-Loss-Ratio and the β-diagram and thereby extends the possibilities of the Gain-Loss-Ratio approach by Bernardo and Ledoit (2000). The main drawbacks of the GLR are exposed by suitable examples and it is shown how the SGLR can overcome them. Additionally, the β-diagram allows to highlight the distribution of attractiveness of an asset/portfolio to a certain investor. The SGLR further has many desirable properties as a performance measure, e.g. it has a strong economic meaning and fulfills all requirements of an acceptability index.

Eventually, it is shown how the SGLR can be used to calculate price intervals on incomplete markets. A topic of further research might be the estimation of the SGLR in specific practical occasions.

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A. Auxiliary Definitions and Propositions

An ordinary market model should be a parsimonious map of a financial market. We assume that the market is observed at two points in time. First, at time 0, money is invested, and at the second point in time, T, the total payout is taken. We assume a market where the underlyings evolve in continuous time as semimartingales and allow for very general time-continuous trading strategies, but in the end we reduce it to an one-period model where we pay the price for the investment today and get the payout X_i at time T investing in strategy i. Most of the assumptions are taken from Biagini and Pinar (2013) section two, which allows to use their duality result in the most central proof of this paper. The formal definition for an *ordinary market model* in this paper is given by:

Definition A.1. *(ordinary market model):*
Let $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be an appropriate filtered probability space, $(S_t)_{0\leq t\leq T}$ a $\mathbb{R}^d$-valued semimartingale such that $(\sup_{t\leq T}|S_t|) \in \mathcal{L}^1$, I an Index-set representing the investment opportunities with payouts

$$\{X_i : i \in I\} = \{X_i \in \mathcal{L}^2 | X_i = \int_0^T \xi_t dS_t \text{ , where } \xi \text{ is simple, predictable, bounded and S-integrable } \}$$

at time T.[27] Let $m \in \mathcal{L}^2$ be a positive stochastic discount factor and $p(X_i) = E[X_i\, m]$ the corresponding price function. The tupel $((\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P), (X_i)_{i\in I}, p)$ is then called ordinary market model.

For the proof of theorem II.1 the assumption

$$\frac{\int_{A_n} m^2\, dP}{(\int_{A_n} m\, dP)^2}(\int_{A_n} M^*\, dP)^2 \overset{n\to\infty}{\longrightarrow} 0 \tag{12}$$

is made. This assumption might not be very intuitive. Therefore the following proposition shows that for cases where the benchmark SDF is bounded, the assumption is fulfilled when the price generating SDF is sufficiently convex in the neighborhood of zero.[28]

Proposition A.1. *Let $M^* \in SDF^+$ with $0 < M^* \leq b$ and $m \in SDF^+$ continuous with*

$$\exists x_0 : \forall 0 < x \leq x_0 : F^m(x) := \int_0^x f^m(t)\, dt \leq c\, x^2, \tag{13}$$

where f^m is the density function of m and $c > 0$. Then the following holds:

$$\frac{\int_{A_n} m^2\, dP}{(\int_{A_n} m\, dP)^2}(\int_{A_n} M^*\, dP)^2 \overset{n\to\infty}{\to} 0,$$

where $A_n := \{\frac{m}{M^*} \leq \frac{1}{n}\}$.

[27] Hence in this set are all payouts at T of buy and hold strategies on S over finitely many trading dates (cp. Biagini and Pinar (2013) chapter 2.1.).

[28] E.g. the SDF in the Black-Scholes model in equation (2) fulfills this property.

Proof.

$$\frac{\int\limits_{A_n} m^2\, dP}{(\int\limits_{A_n} m\, dP)^2}(\int\limits_{A_n} M^*\, dP)^2 \leq \frac{(\frac{1}{n}b)^2 P(A_n)(b^2(P(A_n))^2)}{(\int\limits_0^{(F^m)^{-1}(P(A_n))} x f^m(x)dx)^2}$$

$$\overset{(13)}{\leq} \frac{1}{n^2}\frac{b^4(P(A_n))^3}{(\int\limits_0^{(F^m)^{-1}(P(A_n))} \frac{1}{\sqrt{c}}(F^m(x))^{\frac{1}{2}} f^m(x)dx)^2}$$

$$= \frac{1}{n^2}\frac{b^4(P(A_n))^3}{\left[\frac{2}{3}\frac{1}{\sqrt{c}}(F^m(x))^{\frac{3}{2}}\right]_0^{(F^m)^{-1}(P(A_n))}}$$

$$= \frac{1}{n^2}\frac{b^4(P(A_n))^3}{\frac{2}{3}\frac{1}{\sqrt{c}}(P(A_n))^3}$$

$$= \frac{1}{n^2}\frac{b^4}{\frac{2}{3}\frac{1}{\sqrt{c}}} \overset{n\to\infty}{\longrightarrow} 0$$

□

Assumption (12) ensures that the SGLR is finite. In the proof of theorem II.1 a class of SDFs $\{\underline{\overline{M}}^{\beta^*} : \beta^* > 0\}$ is introduced. For the validity of the proof it is important to show that for a given $\beta > 0$ there is a β^* such that $\beta > \beta^* > 0$ and $\underline{\overline{M}}^{\beta^*} \in SDF^+_\beta(M^*)$. This is done with the next lemma.

Lemma A.1. *Given the situation of definition II.1 and*

$$\frac{\int\limits_{A_n} m^2\, dP}{(\int\limits_{A_n} m\, dP)^2}(\int\limits_{A_n} M^*\, dP)^2 \overset{n\to\infty}{\to} 0\ ,\ with\ A_n := \{\frac{m}{M^*} \leq \frac{1}{n}\}, \tag{14}$$

the following holds true:

$$\exists \beta^* > 0 : (\beta^* < \beta\ \wedge\ \underline{\overline{M}}^{\beta^*} \in SDF^+_\beta(M^*))$$

Proof. Let $\beta^* < \beta$.

1. $P(\underline{\overline{M}}^{\beta^*} = M^*) \leq 1 - \beta$ √
2.

$$\begin{aligned} Var(\underline{\overline{M}}^{\beta^*}) &= E((\underline{\overline{M}}^{\beta^*})^2) - \overbrace{E(\underline{\overline{M}}^{\beta^*})^2}^{1} \\ &= E((\mathbb{1}_{\Omega\setminus(\underline{C}(\beta^*)\cup\overline{C}(\beta^*))}M^*)^2) + \underbrace{E(\mathbb{1}_{\underline{C}(\beta^*)}m^2\underline{c}(\beta^*)^2)}_{\overset{(14)}{\Longrightarrow}\overset{\beta^*\to 0}{\longrightarrow}0} + \underbrace{E(\mathbb{1}_{\overline{C}(\beta^*)}m^2\overline{c}(\beta^*)^2)}_{\overset{\beta^*\to 0}{\longrightarrow}0} - 1 \\ &\overset{\beta^*\to 0}{\longrightarrow} Var(\underline{\overline{M}}^{\beta^*}) \\ &< (1+\beta)Var(\underline{\overline{M}}^{\beta^*})\ \surd \end{aligned}$$

□

B. Graphical illustration of an element of $SDF_{\beta}^{+}(M^*)$

Let X be a normally distributed random variable simulating payouts at time T and let $M^* \equiv 1$ be the (risk-neutral) benchmark SDF. In the class $SDF_{0.01}^{+}(M^*)$ there are many SDFs that imply a smaller GLR than the one concerning the risk-neutral SDF. We want to take a look at a SDF $M' \in SDF_{0.01}^{+}(M^*)$ that produces a GLR close to $SGLR_{0.01}^{M^*}(X)$. $SGLR_{0.01}^{M^*}(X)$ is the infimum over all GLRs implied by SDFs of the class $SDF_{0.01}^{+}(M^*)$. These SDFs differ from M^* only in 1% of the states. We choose a SDF M' such that the implied GLR decreases strongly. Therefore it must maximally both increase its denominator and reduce its enumerator respecting the variance and the positivity constraints. Hence the SDF M^* is changed in states of the tail of X. In figure 3 a plot of $XM^* = X$ (black line), of XM' (gray circles), of $M^* = 1$ (brown line) and M' (coral circles) is given. The plots are rescaled and shifted to qualitatively visualize the effect. The difference between M^*X and $M'X$ is given in the states of the tail of X and creates a higher expected discounted loss and a lower expected discounted gain resulting in a decreased GLR concerning M' in comparison to the GLR induced by $M^* \equiv 1$.

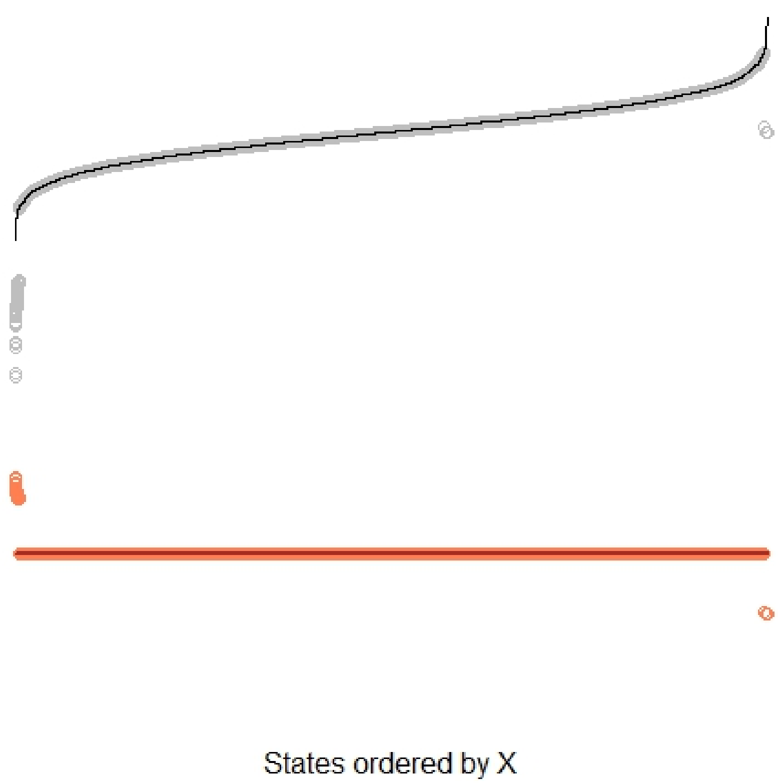

Figure 3: *Discounted payouts: $XM^* = X$ (black line), XM' (gray circles) and corresponding SDFs: $M^* = 1$ (brown line), M' (coral circles)*

C. (S)GLR of example portfolios

I. Example of a portfolio with an undefined GLR

Assume a Black-Scholes Model, i.e. a market with a stock and a money market account, where the price dynamics of the stock is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a standard Wiener process and $\sigma, \mu \in \mathbb{R}^+$. Further assume that binary options are traded. The Black-Scholes-formula for a binary call $call_b(K)$ with strike K is:[29]

$$price(call_b(K)) = e^{-rT}\Phi(d),$$
$$\text{with } d = \frac{ln\frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and Φ as the cumulative distribution function of the normal distribution. For the sake of simplicity we assume $S_0 = \sigma = T = 1$. The price of a binary call is then given by:

$$price(call_b(K)) = e^{-rT}\Phi(r - ln(K) - \frac{1}{2})$$

Let $Prtf_Z$ denote the payout of a portfolio that consists of $(-1)^n \frac{1}{ln(n)price(call_b(n))}$ digital call options for $n \in \mathbb{N} \setminus \{1\}$, where a negative number means a corresponding short position. The price of this portfolio is given by:

$$price(Prtf_Z) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{ln(n)price(call_b(n))} price(call_b(n)) = a < \infty$$

Here the Leibniz criterion ensures that the sum is finite. Whereas the expected gain and the expected loss do diverge:

$$E^+[Prtf_Z] = E^-[Prtf_Z] = \infty$$

Proof. Define $b_n := \frac{1}{ln(n)price(call_b(K))} P(S_T > n)$. Then b_n goes to ∞ as n goes to ∞:

$$\begin{aligned} b_n &= \frac{1 - \Phi(ln(n) - \mu + \frac{1}{2})}{ln(n)price(call_b(n))} \\ &= \frac{\Phi(-ln(n) + \mu - \frac{1}{2})}{ln(n)e^{-r}\Phi(r - ln(n) - \frac{1}{2})} \end{aligned}$$

[29]This option pays out one unit of cash if the stock price is above K at maturity T and nothing else.

For the limit of $\lim_{n\to\infty} b_n$ the following holds

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{\Phi(-ln(n)+\mu-\frac{1}{2})}{ln(n)e^{-r}\Phi(r-ln(n)-\frac{1}{2})} \\
&= \lim_{n\to\infty} \frac{\Phi(-n+\mu-\frac{1}{2})}{e^{-r}\Phi(r-n-\frac{1}{2})n} \\
&\overset{\star}{=} \lim_{n\to\infty} \frac{exp(-\frac{1}{2}(-n+\mu-\frac{1}{2})^2)}{e^{-r}(exp(-\frac{1}{2}(r-n-\frac{1}{2})^2)n+\Phi(r-n-\frac{1}{2}))} \\
&\overset{\star}{=} \lim_{n\to\infty} e^r \frac{exp(-\frac{1}{2}(-n+\mu-\frac{1}{2})^2)(-n+\mu-\frac{1}{2})}{exp(-\frac{1}{2}(r-n-\frac{1}{2})^2)(2+n(r-n-\frac{1}{2}))n} \\
&= \lim_{n\to\infty} exp(n(\mu-r)+(r-\frac{1}{2})^2+(\mu-\frac{1}{2})^2)\frac{n-\mu+\frac{1}{2}}{(2+n(r-n-\frac{1}{2}))n}e^r \\
&= \infty
\end{aligned}$$

$\star$ is justified by the law of de l'Hospital knowing that the relevant limit exists.
Furthermore there is a $N \in \mathbb{N}$ such that b_n is strictly monotonely increasing for $n > N$. Hence $E^+[Prtf_Z] = \sum_{n=2}^{\infty}[(-1)^n b_n - price(call_b(n))]^+ = \infty$
and $E^-[Prtf_Z] = \sum_{n=2}^{\infty}[(-1)^n b_n - price(call_b(n))]^- = \infty$ □

In contrast the SGLR of the portfolio is finite.[30]

II. Example of a portfolio with tail dominated performance

Consider a Black-Scholes model with $S_0 = 1$, $T = 1$, $r = 0.02$, $\mu = 0.06$ and $\sigma = 1$. $Prtf_A$ is given by four plain-vanilla call options with strikes 1, 2, 3 und 4. The corresponding β-diagram is given in figure 1. The calculation is done via a Monte-Carlo method using the procedure described in section I of appendix D to compute the empirical SGLR.
Calculations and plot are done with the following R-Code:[31]

```
cOut <-function(K,S){
  return(max(S-K,0))
}
cPrice <- function (K){
  d1 <- ( log(1/K) + (1/2) )
  d2 <- d1 -1
  return(pnorm(d1)-K*pnorm(d2) )
}
X<-rep(0,10000)

pb <- txtProgressBar(min=0, max=100,style=3)
betadia1<-rep(0,100)
S<-exp(0.06+rnorm(10000))
```

[30] By changing the benchmark SDF in a way that the most extreme positive payouts are valued arbitrarily close to zero the enumerator of the SGLR becomes finite.
[31] It is assumed that the function `empSGLR` of the next section is available.

```
for(j in 1:100){
  for(i in 1:10000){
    X[i]<-max(S[i]-1,0)+max(S[i]-2,0)+max(S[i]-3,0)+max(S[i]-4,0)
-cPrice(1)-cPrice(2)-cPrice(3)-cPrice(4)
  }
  obj1<-empSGLR(X,0.0005*j,0.00001)
  betadia1[j]<-obj1[1]
  setTxtProgressBar(pb, j)
}
plot(0.0005*(1:100),betadia1,type="l",lwd=3,xlab="beta",ylab="SGLR")
```

III. Example of two portfolios with crossing β-diagrams

Consider the situation of the last subsection. Now add another Portfolio $Prtf_B$. It consists of the same call-options, but with more weight in the call options with strike 4 and less in the one with strike 3. Calculations and plot are done with the following R-Code:[32]

```
cOut <-function(K,S){
  return(max(S-K,0))
}
cPrice <- function (K){
  d1 <- ( log(1/K) + (1/2) )
  d2 <- d1 -1
  return(pnorm(d1)-K*pnorm(d2) )
}
X<-rep(0,10000)
Y<-X

pb <- txtProgressBar(min=0, max=100,style=3)
betadia1<-rep(0,100)
betadia2<-rep(0,100)
S<-exp(0.06+rnorm(10000))
for(j in 1:100){
  for(i in 1:10000){
    X[i]<-max(S[i]-1,0)+max(S[i]-2,0)+max(S[i]-3,0)+max(S[i]-4,0)
-cPrice(1)-cPrice(2)-cPrice(3)-cPrice(4)
    Y[i]<-max(S[i]-1,0)+max(S[i]-2,0)+0.1*max(S[i]-3,0)+5*max(S[i]-4,0)
-cPrice(1)-cPrice(2)-0.1*cPrice(3)-5*cPrice(4)
  }
  obj1<-empSGLR(X,0.0005*j,0.00001)
  obj2<-empSGLR(Y,0.0005*j,0.00001)
  betadia1[j]<-obj1[1]
  betadia2[j]<-obj2[1]
  setTxtProgressBar(pb, j)
}
plot(0.0005*(1:100),betadia1,type="l",lwd=3,xlab="beta",ylab="SGLR")
points(0.0005*(1:100),betadia2,type="l",col="gray",lwd=3)
```

[32]It is assumed that the function `empSGLR` of the next section is available.

D. Calculation of price intervals with the SGLR

I. Calculation of the empirical SGLR for a risk-neutral benchmark investor

Calculating the SGLR generally means solving a non-linear optimization problem with two non-linear constraints. In the case of a risk-neutral investor the problem becomes easier. The following R-code implements an estimation algorithm of the SGLR of an asset X concerning a risk-neutral investor based on a sample of n payouts X_i.
The GLR will change the most when changing the benchmark SDF in the states of the tails of X. It is in general not clear which states must be changed to get the minimal value under the given constraints.
Therefore the minimum over all GLRs resulting from the changed benchmark SDFs of all combinations of $\beta \cdot n$ changed states is taken.[33]
For a fixed combination of changeable states the bottleneck is the variance. Therefore changes in the SDF should be done for states where the change of GLR per change of variance is largest. Furthermore a positive change in one state must be combined with a negative change in another state to preserve the SDF properties.[34] This is done here successively by the following steps:

1. The change of GLR per change of variance resulting from a change of the SDF is calculated for each state.
2. The SDF is decreased slightly at the state of the highest 'change of GLR'-'change of variance'-ratio.
3. The SDF is increased by the same amount at the state of the lowest (e.g. negative) 'change of GLR'-'change of variance'-ratio.
4. The first three steps are repeated until the variance of the SDF reaches β.

In R this can be implemented by the following code:

```
#The empirical SGLR for a risk-neutral benchmark investor
set.seed(1)
###
#functions

Gain <- function(X,M){
#Calculates the empirical Gain of X concerning M
  tempg <- sum((X>0)*(X*M))
  return(tempg)
}

Loss <- function(X,M){
#Calculates the empirical Loss of X concerning M
  templ <- -sum((X<0)*(X*M))
  return(templ)
}

Glr <- function(X,M){
#Calculates the empirical GLR of X concerning M
  tempglr <- Gain(X,M)/Loss(X,M)
```

[33] The changed states are still those of the tails of X, but the changes are allowed to be unsymmetrical.
[34] I.e. $E[M] = 1$. Furthermore the changes must be done in a way that $M > 0$.

```
  return(tempglr)
}

deltaProVar<- function(M,delta){
#Calculates the approx. max. growth of a state
#when increasing the variance by delta for each state
  tempdpv<- (-1*(M>1)+(M<1)*1)*(M-1)+sqrt((M-1)^2+length(M)*delta)
  return(tempdpv)
}

GlrProVarPos <- function(gain,loss,X,M,delta){
#Calculates the approx. gain of GLR by increasing the state
#such that the variance rises by delta for each state
  glr<-gain/loss
  tempgpvp <- (X>0)*(gain+X*deltaProVar(M,delta))/(loss*glr)+(X<0)*(gain)/
((loss+X*deltaProVar(M,delta))*glr)
  return(tempgpvp)
}

UrM <- function(beta0,beta1,n){
#changes the benchmark SDF on beta0 states:
#it increases beta1 of them and decreases the rest slightly to initialize the changed SDF
  M <- rep(1,n)
  beta2 <- beta0-beta1
  k<-round(n*beta1)
  j<-round(n*beta2)
  M[1:k]<-rep((1+(0.001/(k))),k)
  M[n:((n+1)-j)]<-rep(1-(0.001/(j)),j)
  return(M)
}

empSGLRaux <- function(X,betta,betta1,delta){
#calculates the empSGLR under the assumption
#that betta1 or less states must have an increased SDF while the others will be decreased.
  M<-UrM(betta,betta1,length(X))
  X <- X[order(X)]
  tempvar<- var(M)
  n<-length(X)
  M0 <- rep(1,n)
  while(tempvar < (betta)){
    tempW <- GlrProVarPos(Gain(X,M), Loss(X,M),X,M,delta)
    #print(betta1)
    if(max((X>0)*(M!=M0)*(M>0))==0){
      if (betta1-1/n>0){
        return(empSGLRaux(X,betta,betta1-1/n,delta))}
      tempvar<-betta
    }else{
      tempmax <- order(tempW*(X>0)*(M!=M0)*(M>0))[n]
      tempmin <-order(tempW*(X<0)*(M!=M0)*(M>0))[n]
      tempdelta <- deltaProVar(M,delta)[tempmax]
      M[tempmax]<- M[tempmax]- tempdelta
      M[tempmin]<- M[tempmin]+ tempdelta
```

```
      tempvar <- var(M)
    }}
  tempret <- c(Glr(X,M),Glr(X,rep(1,length(X))),M)
  return(tempret)
}

empSGLR<-function(X,betta,delta){
#Iterates over the number of increased/decreased SDF-states
#and calculates the empirical SDF for the risk-neutral benchmark investor
#and gives the SGLR, the GLR and the changed SDF back (must be inversed where required)
  n<-length(X)
  tempemp1<-empSGLRaux(X,betta,(1/n),delta)
  tempemp2<-empSGLRaux(X,betta,(2/n),delta)
  tempmin<-tempemp1
  i<-3
  while((tempemp2[1]!=tempemp1[1])){
    tempemp1<-tempemp2
    tempemp2<-empSGLRaux(X,betta,(i/n),delta)
    if (tempemp2[1]<tempmin[1]){tempmin<-tempemp2}
    i<-i+1
  }
  return(tempmin)
}
```

II. R-code for the calculation of the price-intervals of a Call option without intertemporal trading

It is assumed that the function `empSGLR` of the last subsection is available.

```
#packages
library(e1071)
library(MASS)
set.seed(123)
#const
#...of the price dynamic
S0 <- 100       #start price asset
K<-150          #strike price of the Option
lfzt <- 1       #time horizon in years
r <- 1.05       #one year linear interest rate
rs <- 1.05      #one year drift asset
a <- 1          #stochastic vola and jumps yes=1 no=0
gamma <- 0.03 #intensity of the poisson process
mu <- 2         #jump vola parameter
kappa <- 4      #mean reversion speed of the stochastic volatilty
theta <- 0.2  #mean reversion level of the stochastic volatilty
vola0 <- 0.2  #start value of the volatility
b <- 0.5        #coefficient that controls for affinity, i.e. for b=0.5 the model is affine
sigma <- 0.1  #volatility of volatility
rho <- -0.5   #correlation coefficient between W1 and B1

#...of exactness (grid seize)
N <-1000        # number of simulations
n <- 100        # steps of the price trajectory
```

```
N1 <-11     # steps for the weight-grid for the different possible portfolios
# build from Option and Stock
N2 <- 100     # steps for the Price-grid of the option,
# with which the optimization under the constrain SGLR<L is made

#...of the SGLR
beta <- 0.02  #subsistence level of the SGLR
L <- 1.2      #The (assumed) beta-SGLR of the market.


#var
ST <- rep(NA,N)#Price of the Stock at time T (will be simulated N times)


#functions
stocksim <- function(r,a,gamma,mu,kappa,vola0,b,sigma,rho,N){
  #The function simulates N end-prices ST
  #by simulation of the trajectory with a n-step Euler-Maruyama approach
  ###
  pb <- txtProgressBar(min=0, max=N,style=3) #creates a progressbar
  #local variables:
  STtemp <- rep(NA,N)
  Vt<-rep(NA,n)
  Vt[1] <- vola0
  for (j in 1:N){
    #simulation of the used wiener processes:
    Z <- rwiener(end=lfzt,  frequency = n)
    W <- (sqrt(1-abs(rho)))*rwiener(end=lfzt,  frequency =n)+sqrt(abs(rho))*Z
    B <- (sqrt(1-abs(rho[1])))*rwiener(end=lfzt,  frequency = n)+sign(rho)*sqrt(abs(rho))*Z
    St <- S0
    #simulation of a trajectory with n-steps:
    for (i in 2:n){
      St <-abs( St+St*(1-rs)/n+sqrt(Vt[i-1])*St*(W[i]-W[i-1])+
a*((-gamma*mu*St)/n+(exp(rnorm(1,0,mu))-1)*(rpois(1,gamma))))
      Vt[i] <- Vt[i-1]+kappa*(theta-Vt[i-1])/n+sigma*(Vt[i-1])^b*(B[i]-B[i-1])
    }
    STtemp[j] <- St
    setTxtProgressBar(pb, j)
  }
  return(STtemp)
}


xfunc<- function(S0,ST,C,beta){
  #This function calculates the "empirical beta-SGLR" of a market consisting
#of a Stock with start price S0 and possible end-prices ST
  #and a Call option with price C (ignoring short positions in the Call)
  sg <- rep(NA,N1)
  for (i in 1:N1) {
    w0<- (i-((N1-1)/2+1))/1000
    w1<- 1-w0
    x<- (w0*(ST-S0/r)+w1*((ST-K)*((ST-K)>0)-C/r))
```

```
    sg[i]<-empSGLR(x,beta,0.0001)[1]
  }
  return(max(sg))
}

xfuncm<- function(S0,ST,C,beta){
  #This function calculates the "empirical beta-SGLR" of a market
#consisting of a Stock with start price S0
  #and possible end-prices ST and a short-position in a Call option with price C
#(ignoring long positions in the Call)
  sg <- rep(NA,N1)
  for (i in 1:N1) {
    w0<- (i-((N1-1)/2+1))/1000
    w1<- 1-w0
    x<- (w0*(ST-S0/r)-w1*((ST-K)*((ST-K)>0)-C/r))
    sg[i]<-empSGLR(x,beta,0.0001)[1]
  }
  return(max(sg))
}

boundary <- function(S0,ST,N2,beta,L){
  #this function calculates the bounds of the call price implied by a beta-SGLR < L and the
  #empirical distribution of ST

  #calculation of the lower bound
  sp <- L+1
  j<-1
  while (sp  >L ) {
    sp <-xfunc(S0,ST,(j*10/N2),beta)
    j<-j+1
  }
  untereSchranke <- j/N2*10

  #calculation of the upper bound
  sp <- L-1
  j<-1
  while (sp  < L ) {
    sp <-xfuncm(S0,ST,(j*10/N2),beta)
    j<-j+1
  }
  obereSchranke <- j/N2*10+untereSchranke
  #creation of an output
  Ausgabe <- c("Untere Schranke:",untereSchranke,"Obere Schranke:",obereSchranke)
  return(c(untereSchranke,obereSchranke))
}
#####
#"The Program" calculates the price intervals implied by the assumed beta-SGLR of the market
# and the empirical distribution of the simulated stock-price
#simulation step:
ST<-stocksim(r,a,gamma,mu,kappa,vola0,b,sigma,rho,N)
truehist(ST)
boundary(S0,ST,N2,beta,L)
```