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An exact pricing formula for European call options on zero-coupon bonds in the run-up to a currency union

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Abstract. In this paper we analyze the dynamics of zero-coupon bond options in a situation in which two open economies plan to enter a currency union in the future. More precisely, we make use of recent theoretical work on the continuous-time dynamics of interest-rate differentials between the economies involved and derive a closed-form pricing formula for a European call option on zero-coupon bonds. In a Monte-Carlo simulation study we show that significant option-pricing errors can occur when the key features of interest-rate dynamics during the run-up to the currency union are ignored.

JEL-classification codes: G12, G13, G15, E42, F37

Key words: Interest-rate dynamics; valuation of interest-rate options; currency union

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1 Introduction

Closed-form solutions for European options on pure discount bonds and on discount bond portfolios have been established in a classical option-pricing framework by Jamshidian (1989). Using Vasicek's (1977) mean-reverting Gaussian interest-rate model and assuming that the term structure is completely determined by the value of the instantaneous interest rate, the author derives a closed-form Black-Scholes-type pricing formula. In this paper we leave this classical option-pricing framework and modify Jamshidian's (1989) results by taking into account that a country's interest-rate dynamics—which is relevant to option-pricing—may be closely linked to the interest rates of the partner countries via the current exchange-rate system.

Two alternative exchange-rate arrangements under which the interest rates of the countries involved are intimately connected to each other are well-documented in the economic literature. The first arrangement is a so-called exchange-rate target zone as introduced by Krugman (1991). The dynamic interrelationships between the participating countries' interest rates (of arbitrary terms) are derived in Svensson (1991a, 1991b). The second exchange-rate arrangement is represented by the time period prior to the fixing of a currently floating exchange rate on a given future date at a publicly announced fixing parity. In a stylized model, Wilfling (2003) derives the term structure of the bilateral interest-rate differentials under such an exchange-rate regime thus providing dynamic equations for the link governing the interest rates in both countries.

Owing to its political topicality, this paper focusses on the second of the just-mentioned exchange-rate regimes. In practice, the introduction of a common currency is typically initiated by a switch in exchange-rate system from (more or less) floating exchange rates to completely fixed rates. For example, the introduction of the euro among the member countries of the European Monetary Union (EMU) was implemented by the irreversible fixing of the EMU countries' bilateral exchange rates at their respective central parities from the European Exchange Rate Mechanism (ERM) from 1 January 1999 onwards. Since then, the same exchange-rate fixing procedure has been applied to all later EMU entrants and it is very likely that future EMU accession countries will also enter the currency union at conversion rates equal to their ERM central parities vis-à-vis the euro.

Up to date, the EMU consists of 16 countries including the large economies of France, Germany, Italy and Spain. There are, however, several other major European economies that have not yet become EMU members, but are likely to adopt the euro in the future (like Poland, Sweden and the UK). It is the financial market participants operating in these future EMU accession countries to whom our closed-form formulas for zero-coupon bond options established below should be of particular relevance.

The remainder of the paper is organized as follows. Section 2 reviews some previous results on exchange-rate dynamics and on international interest-rate differentials in the run-up to a currency union. Based on these results we derive the interest-rate dynamics crucial to our option-pricing problem. In Section 3 we first value zero-coupon bonds under the new interest-rate dynamics and then value European call options on these pure discount bonds. In Section 4 we conduct a Monte-Carlo simulation study in order to assess the validity of our option-pricing formula. Section 5 offers some concluding comments.

2 Previous results on exchange-rate and interest-rate dynamics

In what follows we consider a world with two open economies under perfect capital mobility and assume the domestic economy to be small. In this general setting, let the political authorities of the two economies decide to create a currency union in the future. On the analogy of Stage III of EMU, the authorities therefore announce at date t_A to irreversibly fix the exchange rate from the future date t_S onwards (i.e. $t_A < t_S$) at the specific exchange-rate parity s.

The exchange-rate dynamics under such a time-contingent switch in exchange-rate regime has been characterized in the literature by various authors on the basis of the well-known monetary exchange-rate model with flexible prices (see, among others, Sutherland, 1995; De Grauwe et al., 1999; Wilfling and Maennig, 2001). In this continuous-time equilibrium model with rational expectations, the logarithmic spot exchange rate at time $t, x(t)$, equals the sum of two components: (a) an exogenously given 'macroeconomic fundamental variable' $k(t)$, and (b) a speculative term representing the expected (instantaneous) rate of change in the nominal exchange rate:

$$x(t) = k(t) + \alpha \cdot \frac{E[dx(t)|\phi_t]}{dt}, \alpha > 0. \tag{1}$$

In Eq. (1), $E[\cdot|\phi_t]$ denotes the expectation operator conditional on the information set ϕ_t which contains all information available to market participants at time t. The parameter α represents the semi-elasticity of money demand with respect to the instantaneous interest rate. Alternatively, α may simply be interpreted as a parameter weighting the fundamental component against the speculative motives for currency valuation.

In the monetary flex-price model the fundamental $k(t)$ represents an aggregate of given macroeconomic variables (such as the domestic and foreign money supplies and outputs) as well as stochastic shocks to money demand. Via the domestic and foreign money supplies $k(t)$ is under direct control of the two central banks involved and, prior to the fixing-date t_S, $k(t)$ should follow an appropriate continuous-time stochastic process. In this paper, we model the evolution of $k(t)$ over time (up to t_S) by a driftless Brownian motion with stochastic differential representation

$$dk(t) = \tilde{\sigma} \cdot d\bar{W}(t), t < t_S, \tag{2}$$

with (constant) infinitesimal standard deviation $\tilde{\sigma} > 0$ and $d\bar{W}(t)$ the increment of standard Wiener process. The driftless Brownian motion is particularly adequate when modeling a situation in which the central banks refrain from intervening in the foreign exchange market. Thus, modeling the fundamental $k(t)$ as in Eq. (2) is consistent with assuming a pure free-float exchange-rate regime prior to the currency union.[1]

Given the specification (2) of the fundamental process, the general law of exchange-rate dynamics in Eq. (1) constitutes a stochastic differential equation. This can be solved by standard techniques and the imposition of adequate economic constraints, which appropriately reflect the anticipations of foreign exchange market participants with regard to the entrance of both economies into the currency union on date t_S at the parity s. Ruling out currency-arbitrage opportunities at the moment of transition into the currency union (i.e. imposing the condition $\lim_{t \to t_S} x(t) = s$ with probability 1) it is straightforward to check that the (bubble-free) solution to Eq. (1) is given by

$$x(t) = \begin{cases} k(t) & \text{for } t < t_A \\ k(t) + e^{(t-t_S)/\alpha} \cdot [s - k(t)] & \text{for } t \in [t_A, t_S) \\ s & \text{for } t \geq t_S \end{cases} . \tag{3}$$

Next, we establish the interest-rate dynamics in the two economies by adopting the model set-up presented in Wilfling (2003). Let $P(t,T)$ denote the price at time t of a domestic zero-coupon bond maturing at time T, $t \leq T$, with unit maturity value $P(T,T) = 1$ and define $P^*(t,T)$ to be the analogous price of a foreign-currency discount bond. Furthermore, let us denote the domestic and the foreign instantaneous

[1] More interventionist exchange-rate policies prior to the currency union can be modeled by specifying alternative driving processes for the fundamental $k(t)$. Sondermann et al. (2010), for example, model (a) an exchange-rate system of managed floating and (b) a system of continuously increasing interventionst activity towards the entrance into the currency union by letting the fundamental $k(t)$ follow an Ornstein-Uhlenbeck process and a scaled Brownian bridge, respectively.

short rates at time t by $r(t)$ and $r^*(t)$, respectively, and suppose that the small domestic economy cannot affect the foreign short rate by economic policy, but has to accept $r^*(t)$ as exogenously given. We further assume (a) perfect international capital mobility, and (b) that international investors consider the domestic and the foreign discount bonds as perfect substitutes. Under this scenario the following form of the uncovered interest parity condition should hold among the instantaneous short rates at all points in time:[2]

$$\text{SRD}(t) \equiv r(t) - r^*(t) = \frac{E[dx(t)|\phi_t]}{dt}. \tag{4}$$

The exchange-rate path (3) plus the uncovered interest parity condition (4) now allow us to represent the short-rate differential SRD(t) in closed form. To this end, we apply Ito's lemma to the exchange-rate path (3) which yields the stochastic differential $dx(t)$. After taking conditional expectations and dividing by dt we obtain the short-rate differential during the run-up to the currency union:

$$\text{SRD}(t) = r(t) - r^*(t) = \begin{cases} 0 & \text{for } t < t_A \\ e^{(t-t_S)/\alpha} \cdot \dfrac{s - k(t)}{\alpha} & \text{for } t \in [t_A, t_S) \\ 0 & \text{for } t \geq t_S \end{cases} . \tag{5}$$

Finally, we follow Vasicek (1977) and let the exogenously given foreign short rate $r^*(t)$ evolve according to a mean-reverting Ornstein-Uhlenbeck process with stochastic differential

$$dr^*(t) = b(c - r^*(t))dt + \sigma dW_1(t), \tag{6}$$

where b, c, σ are positive constants and $W_1(t)$ denotes a standard Wiener process. Given the initial value $r_0^* \equiv r^*(0)$ the solution to Eq. (6) is known to be

$$r^*(t) = (r_0^* - c)e^{-bt} + c + A(t) \tag{7}$$

with $A(t)$ defined as

$$A(t) \equiv \sigma e^{-bt} \int_0^t e^{bs} dW_1(s).$$

Inserting Eq. (7) into Eq. (5) and taking as given the initial value $k_0 \equiv k(0)$ for the

[2] We understand the uncovered interest parity as an equilibrium condition in the sense that the foreign exchange market is in equilibrium when deposits of all currencies offer the same expected rate of return (with respect to the country-specific short rates). This is (approximately) the case if the short-rate differential equals the expected instantaneous rate of change in the exchange rate.

fundamental process (2), we obtain the domestic short-rate process:

$$r(t)=\begin{cases}(r_0^*-c)e^{-bt}+c+A(t) & \text{for } t<t_A\\ (r_0^*-c)e^{-bt}+c+A(t)+e^{(t-t_S)/\alpha}\cdot\dfrac{s-k_0-\tilde{\sigma}\bar{W}(t)}{\alpha} & \text{for } t\in[t_A,t_S)\\ (r_0^*-c)e^{-bt}+c+A(t) & \text{for } t\geq t_S\end{cases}. \quad (8)$$

In what follows we assume that the Wiener processes $\bar{W}(t)$ and $W_1(t)$ from the Eqs. (2) and (6) are interrelated by $\bar{W}(t)=\beta W_1(t)+\sqrt{1-\beta^2}W_2(t)$ with $-1\leq\beta\leq 1$ and $W_2(t)$ being an intermediary Wiener process independent of $W_1(t)$. Via this assumption, we allow our driving Wiener processes $\bar{W}(t)$ and $W_1(t)$ to be correlated with constant correlation coefficient β (i.e. $\mathrm{Corr}[\bar{W}(t),W_1(t)]=\beta$ for all t).

3 Bond and option valuation

For the purpose of bond and option valuation, we denote the (risk neutral) martingale measure by Q. Following the well-established martingale modeling approach, we specify our short-rate dynamics from Eq. (8) under Q.[3] In Section 3.1 we first value zero-coupon bonds under our Q-dynamics for the short rate and then proceed with the pricing of zero-coupon bond options in Section 3.2.

3.1 Valuation of zero-coupon bonds

The price $P(\theta,T)$ at time θ of a domestic zero-coupon bond maturing at time T is given by the risk-neutral valuation formula

$$P(\theta,T)=E^Q\left[e^{-\int_\theta^T r(t)dt}|\phi_\theta\right] \quad (9)$$

(see for example Björk, 2004, p. 322). To calculate this conditional expectation under Q three distinct cases concerning the dates θ and T have to be distinguished:

Case 1: $\theta<t_A$ or $\theta\geq t_S$.
Case 2: $t_A\leq\theta<t_S$ and $T<t_S$.
Case 3: $t_A\leq\theta<t_S$ and $T\geq t_S$.

Case 1 represents the following two extreme scenarios. (a) If $\theta<t_A$ the prospective currency union has not yet been announced so that financial market participants are currently not aware of the future currency union. (b) If $\theta\geq t_S$ our two economies

[3]For alternative classical models of the Q-dynamics for the short rate $r(t)$ see, among others, Vasicek (1977), Cox et al. (1985), Ho and Lee (1986), Hull and White (1994).

already live in the currency union. In contrast to these two scenarios, the Cases 2 and 3 represent a transitional setting (the so-called interim period) in that for $t_A \leq \theta < t_S$ the currency union has already been announced to financial market participants, but has not yet been implemented. However, according to the Eqs. (3) and (5), the mere announcement of entering a currency union in the future already affects today's exchange-rate as well as today's domestic short-rate dynamics and, consequently, also has an impact on today's pricing of zero-coupon bonds. Moreover, as will become evident below, the exact bond-pricing formula additionally hinges on the question of whether the maturity date T lies before or after the start of the currency union (Case 2 or Case 3).

The calculation of the conditional expectation on the right-hand side of Eq. (9) requires knowledge of the probability distribution of the short rate $r(t)$. In view of Eq. (8) it is straightforward to verify that $\{r(t)\}$ is a Gaussian process and is thus completely characterized in terms of its first and second moments. Setting the present date $\theta = 0$ for ease of notation, we summarize the expectations, variances and covariances of $\{r(t)\}$ in the following lemma.

Lemma 3.1: *The expectations, variances and covariances of the short-rate process $\{r(t)\}$ are given as follows:*

(a) For $t < t_A$ we have

$$\begin{aligned} E[r(t)] &= (r_0^* - c)e^{-bt} + c, \\ \mathrm{Var}[r(t)] &= \frac{\sigma^2}{2b}(1 - e^{-2bt}), \\ \mathrm{Cov}[r(t), r(t')] &= \frac{\sigma^2}{2b} e^{-b(t+t')}(e^{2b\min\{t,t'\}} - 1). \end{aligned}$$

(b) For $t_A \leq t, t' < t_S$ we have

$$\begin{aligned} E[r(t)] &= (r_0^* - c)e^{-bt} + c + e^{\frac{t-t_S}{\alpha}} \frac{s - k_0}{\alpha}, \\ \mathrm{Var}[r(t)] &= \frac{\sigma^2}{2b}(1 - e^{-2bt}) + \frac{\tilde{\sigma}^2}{\alpha^2} e^{\frac{2t-2t_S}{\alpha}} t \\ &\quad - 2\beta \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} e^{-bt} e^{\frac{t-t_S}{\alpha}} \min\{e^{2bt} - 1, t\}, \\ \mathrm{Cov}[r(t), r(t')] &= \frac{\sigma^2}{2b} e^{-b(t+t')}(e^{2b\min\{t,t'\}} - 1) + \frac{\tilde{\sigma}^2}{\alpha^2} e^{\frac{t+t'-2t_S}{\alpha}} \min\{t, t'\} \\ &\quad - \beta \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} e^{-bt'} e^{\frac{t-t_S}{\alpha}} \min\{e^{2bt'} - 1, t\} - \beta \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} e^{-bt} e^{\frac{t'-t_S}{\alpha}} \min\{e^{2bt} - 1, t'\}. \end{aligned}$$

(c) For $t \geq t_S$ we have

$$\begin{aligned} E[r(t)] &= (r_0^* - c)e^{-bt} + c, \\ \mathrm{Var}[r(t)] &= \frac{\sigma^2}{2b}(1 - e^{-2bt}), \\ \mathrm{Cov}[r(t), r(t')] &= \frac{\sigma^2}{2b} e^{-b(t+t')}(e^{2b\min\{t,t'\}} - 1). \end{aligned}$$

Next, we address the integral of the short rate $r(t)$ appearing on the right-hand side of Eq. (9). The following lemma provides helpful insight into the probabilistic nature of this integral. Its proof is sketched in Elliot and Kopp (2005, p. 265).

Lemma 3.2: *Let $\{X(t)\}$ be a Gaussian process with continuous sample paths and mean and covariance functions $m(t) \equiv E[X(t)]$ and $n(t, t') \equiv \mathrm{Cov}[X(t), X(t')]$. Then, the process defined by*

$$Z(t) \equiv \int_0^t X(s)ds$$

is also a Gaussian process with mean and covariance functions given by $\int_0^t m(s)ds$ and $\int_0^t \int_0^{t'} n(u, v)dudv$, respectively.

Lemma 3.2 implies that the process defined by $\int_0^T r(t)dt$ is a Gaussian process. Thus, the random variable $\exp\{-\int_0^T r(t)dt\}$ appearing on the right-hand side of Eq. (9) has a lognormal distribution the expectation of which is uniquely determined by the expectation and the variance of $\int_0^T r(t)dt$. These latter moments follow from Lemma 3.2 and are compiled in the following lemma.

Lemma 3.3: *For the three cases considered above the expectations and variances of $\int_0^T r(t)dt$ are given as follows:*

Case 1: For $\theta = 0 < t_A$ or for $\theta = 0 \geq t_S$ we have

$$\begin{aligned} E\left[\int_0^T r(t)dt\right] &= cT + \frac{r_0^* - c}{b}(1 - e^{-bT}), \\ \mathrm{Var}\left[\int_0^T r(t)dt\right] &= \frac{\sigma^2}{2b^3}(2bT - 3 + 4e^{-bT} - e^{-2bT}). \end{aligned}$$

Case 2: For $t_A \leq \theta = 0 < t_S$ and $T < t_S$ we have

$$
\begin{aligned}
E\left[\int_0^T r(t)dt\right] &= cT + \frac{r_0^* - c}{b}(1 - e^{-bT}) + (s - k_0)(e^{\frac{T-t_S}{\alpha}} - e^{\frac{-t_S}{\alpha}}), \\
\mathrm{Var}\left[\int_0^T r(t)dt\right] &= \frac{\sigma^2}{2b^3}(2bT - 3 + 4e^{-bT} - e^{-2bT}) + \frac{\tilde{\sigma}^2\alpha}{2}e^{\frac{2T-2t_S}{\alpha}}(2\frac{T}{\alpha} - 3 + 4e^{-\frac{T}{\alpha}} - e^{-2\frac{T}{\alpha}}) \\
&\quad - 2\beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{-\frac{t_S}{\alpha}}\int_0^T\int_0^T e^{\frac{v}{\alpha}}e^{-bu}\min\{e^{2bu} - 1, v\}dudv.
\end{aligned}
$$

Case 3: For $t_A \leq \theta = 0 < t_S$ and $T > t_S$ we have

$$
\begin{aligned}
E\left[\int_0^T r(t)dt\right] &= cT + \frac{r_0^* - c}{b}(1 - e^{-bT}) + (s - k_0)(1 - e^{\frac{-t_S}{\alpha}}), \\
\mathrm{Var}\left[\int_0^T r(t)dt\right] &= \frac{\sigma^2}{2b^3}(2bT - 3 + 4e^{-bT} - e^{-2bT}) + \frac{\tilde{\sigma}^2\alpha}{2}(2\frac{t_S}{\alpha} - 3 + 4e^{-\frac{t_S}{\alpha}} - e^{-2\frac{t_S}{\alpha}}) \\
&\quad - 2\beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{-\frac{t_S}{\alpha}}\int_0^{t_S}\int_0^T e^{\frac{v}{\alpha}}e^{-bu}\min\{e^{2bu} - 1, v\}dudv.
\end{aligned}
$$

Finally, we exploit the well-known result that for a normally distributed random variable $X \sim N(\mu, \sigma^2)$ the transformed variable $Y \equiv \exp\{-X\}$ has a lognormal distribution with expected value $E(Y) = \exp\{-\mu + \sigma^2/2\}$. Using this relationship, we are able to calculate the expectation on the right-hand side of Eq. (9) and thus obtain our bond-price formulas which we compile in the following proposition.

Proposition 3.4: *In the run-up to a currency union the price $P(\theta, T)$ at time $\theta = 0$ of a domestic zero-coupon bond maturing at time T is given as follows:*

Case 1: For $\theta = 0 < t_A$ or for $\theta = 0 \geq t_S$ the bond price is given by

$$
P(0,T) = \exp\left\{-cT + \frac{r_0^* - c}{b}(e^{-bT} - 1) + \frac{\sigma^2}{4b^3}(2bT - 3 + 4e^{-bT} - e^{-2bT})\right\}.
$$

Case 2: For $t_A \leq \theta = 0 < t_S$ and $T < t_S$ the bond price is given by

$$
\begin{aligned}
P(0,T) &= \exp\Bigg\{-cT + \frac{r_0^* - c}{b}(e^{-bT} - 1) - (s - k_0)(e^{\frac{T-t_S}{\alpha}} - e^{\frac{-t_S}{\alpha}}) \\
&\quad + \frac{\sigma^2}{4b^3}(2bT - 3 + 4e^{-bT} - e^{-2bT}) + \frac{\tilde{\sigma}^2\alpha}{4}e^{\frac{2T-2t_S}{\alpha}}(2\frac{T}{\alpha} - 3 + 4e^{-\frac{T}{\alpha}} - e^{-2\frac{T}{\alpha}}) \\
&\quad - \beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{-\frac{t_S}{\alpha}}\int_0^T\int_0^T e^{\frac{v}{\alpha}}e^{-bu}\min\{e^{2bu} - 1, v\}dudv\Bigg\}.
\end{aligned}
$$

Case 3: For $t_A \le \theta = 0 < t_S$ and $T \ge t_S$ the bond price is given by

$$
\begin{aligned}
P(0,T) \;=\; \exp\Bigg\{ & -cT + \frac{r_0^* - c}{b}(e^{-bT} - 1) - (s - k_0)(1 - e^{\frac{-t_S}{\alpha}}) \\
& + \frac{\sigma^2}{4b^3}(2bT - 3 + 4e^{-bT} - e^{-2bT}) + \frac{\tilde{\sigma}^2\alpha}{4}(2\frac{t_S}{\alpha} - 3 + 4e^{-\frac{t_S}{\alpha}} - e^{-2\frac{t_S}{\alpha}}) \\
& - \beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{-\frac{t_S}{\alpha}}\int_0^{t_S}\int_0^T e^{\frac{v}{\alpha}}e^{-bu}\min\{e^{2bu} - 1, v\}dudv\Bigg\}.
\end{aligned}
$$

3.2 Valuation of call options on zero-coupon bonds

We now consider a European call option on a zero-coupon bond with maturity date T. Denoting the exercise date of the option by τ ($\tau < T$) and the option's strike price by K, we can write its contract function as $\max\{P(\tau,T) - K, 0\}$ and the risk-neutral valuation formula of the European call option is given by

$$
C(0) = E^Q\left[e^{-\int_0^\tau r(t)dt} \cdot \max\{P(\tau,T) - K, 0\}|\phi_0\right], \tag{10}
$$

where again we have set the current date equal to 0 for ease of notation.

It is important to note here that the bond price $P(\tau,T)$ constitutes a random variable for all exercise dates $\tau > 0$. Thus, the calculation of the expected value on the right-hand side of Eq. (10) requires knowledge of the following three distributions:

(a) the distribution of $\int_0^\tau r(t)dt$,

(b) the distribution of $P(\tau,T)$,

(c) the joint distribution of $\int_0^\tau r(t)dt$ and $P(\tau,T)$.

Since the normal distribution of $\int_0^\tau r(t)dt$ has already been characterized by Lemma 3.3, it remains to find the distributions of the random variable from item (b) and the random vector from item (c). To this end, the following four cases concerning the dates τ and T have to be distinguished:

Case (a): $0 < t_A$ or $0 \ge t_S$.
Case (b): $t_A \le 0 < \tau < T < t_S$.
Case (c): $t_A \le 0 < \tau < t_S \le T$.
Case (d): $t_A \le 0 < t_S \le \tau < T$.

According to Proposition 3.4 we can write the *stochastic* bond prices $P(\tau,T)$ as follows.

Lemma 3.5: *Case (a): For $0 < t_A$ or for $0 \geq t_S$ the bond price can be written as*

$$
\begin{aligned}
P(\tau, T) &= \exp\left\{ -c(T-\tau) + \frac{r^*(\tau) - c}{b}\left[e^{-b(T-\tau)} - 1\right] \right. \\
&\quad \left. + \frac{\sigma^2}{4b^3}\left[2b(T-\tau) - 3 + 4e^{-b(T-\tau)} - e^{-2b(T-\tau)}\right] \right\}.
\end{aligned}
$$

Case (b): For $t_A \leq 0 < \tau < T < t_S$ the bond price can be written as

$$
\begin{aligned}
P(\tau, T) &= \exp\left\{ -c(T-\tau) + \frac{r^*(\tau) - c}{b}\left[e^{-b(T-\tau)} - 1\right] - (s - k(\tau))\left[e^{\frac{T-t_S}{\alpha}} - e^{\frac{\tau-t_S}{\alpha}}\right] \right. \\
&\quad + \frac{\sigma^2}{4b^3}\left[2b(T-\tau) - 3 + 4e^{-b(T-\tau)} - e^{-2b(T-\tau)}\right] \\
&\quad + \frac{\tilde{\sigma}^2\alpha}{4} e^{\frac{2T-2t_S}{\alpha}}\left[2\frac{T-\tau}{\alpha} - 3 + 4e^{-\frac{T-\tau}{\alpha}} - e^{-2\frac{T-\tau}{\alpha}}\right] \\
&\quad \left. - \beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} e^{\frac{\tau-t_S}{\alpha}} \int_0^{T-\tau}\int_0^{T-\tau} e^{\frac{v}{\alpha}} e^{-bu} \min\{e^{2bu} - 1, v\} du dv \right\}.
\end{aligned}
$$

Case (c): For $t_A \leq 0 < \tau < t_S \leq T$ the bond price can be written as

$$
\begin{aligned}
P(\tau, T) &= \exp\left\{ -c(T-\tau) + \frac{r^*(\tau) - c}{b}\left[e^{-b(T-\tau)} - 1\right] - (s - k(\tau))\left[1 - e^{\frac{\tau-t_S}{\alpha}}\right] \right. \\
&\quad + \frac{\sigma^2}{4b^3}\left[2b(T-\tau) - 3 + 4e^{-b(T-\tau)} - e^{-2b(T-\tau)}\right] \\
&\quad + \frac{\tilde{\sigma}^2\alpha}{4}\left[2\frac{t_S-\tau}{\alpha} - 3 + 4e^{-\frac{t_S-\tau}{\alpha}} - e^{-2\frac{t_S-\tau}{\alpha}}\right] \\
&\quad \left. - \beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} e^{\frac{\tau-t_S}{\alpha}} \int_0^{t_S-\tau}\int_0^{T-\tau} e^{\frac{v}{\alpha}} e^{-bu} \min\{e^{2bu} - 1, v\} du dv \right\}.
\end{aligned}
$$

Case (d): For $t_A \leq 0 < t_S \leq \tau < T$ the bond price can be written as

$$
\begin{aligned}
P(\tau, T) &= \exp\left\{ -c(T-\tau) + \frac{r^*(\tau) - c}{b}\left[e^{-b(T-\tau)} - 1\right] \right. \\
&\quad \left. + \frac{\sigma^2}{4b^3}(2b(T-\tau) - 3 + 4e^{-b(T-\tau)} - e^{-2b(T-\tau)}) \right\}.
\end{aligned}
$$

One immediate consequence of Lemma 3.5 is that for each of the four Cases (a) to (d) the required joint distribution of $\int_0^\tau r(t)dt$ and $P(\tau, T)$ is completely characterized in terms of the following joint distributions:

Case (a): $\left(\int_0^\tau r(t)dt, r^*(\tau)\right)$.

Case (b): $\left(\int_0^\tau r(t)dt, r^*(\tau)\left[1 - e^{-b(T-\tau)}\right] + k(\tau)\left[be^{\frac{\tau - t_S}{\alpha}} - be^{\frac{T-t_S}{\alpha}}\right]\right)$.

Case (c): $\left(\int_0^\tau r(t)dt, r^*(\tau)\left[1 - e^{-b(T-\tau)}\right] + k(\tau)\left[be^{\frac{\tau - t_S}{\alpha}} - b\right]\right)$.

Case (d): $\left(\int_0^\tau r(t)dt, r^*(\tau)\right)$.

It is obvious from the preceding section that the random variables $\int_0^\tau r(t)dt$, $r^*(\tau)$ and $k(\tau)$ have normal distributions and that the latter bivariate random vectors all have bivariate normal distributions which are completely characterized in terms of their respective marginal expectations, variances and covariances. Exact expressions for these magnitudes are given in the technical appendix.

From here, we are able to find the joint distribution of $\int_0^\tau r(t)dt$ and $P(\tau, T)$ and thus, ultimately, to calculate the expectation on the right-hand side of Eq. (10). We defer the technical details of this procedure to the appendix. The following Proposition 3.6 summarizes the results by stating price equations for a European call option on zero-coupon bonds in the run-up to a currency union. In these case-specific option-pricing formulas we introduce some new notation. $\Phi(\cdot)$ denotes the standard normal cumulative distribution function, while $\Gamma(b, \alpha, \beta, \sigma, \tilde{\sigma}, \tau, T, t_S)$ is a case-specific parameter-dependent function the intricate structural form of which is given in the Eqs. (A.8) to (A.10) of the appendix. Moreover, the pricing formulas contain the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ which have not yet been defined. As described in the equation blocks (A.2) to (A.5) of the appendix these case-specific auxiliary parameters are certain functions of previously defined parameters.

Proposition 3.6: *In the run-up to a currency union the current price $C(0)$ of a European call option on a zero-coupon bond maturing at time T with strike price K and exercise date τ is given as follows:*

Case (a): For $0 < t_A$ or for $0 \geq t_S$ the option price is given by

$$
\begin{aligned}
C(0) &= P(0,T) \cdot \Phi\left(\frac{y_0 - \left[\mu_2 - \rho\sigma_1\sigma_2 + \frac{\sigma_2^2}{b}(e^{-b(T-\tau)} - 1)\right]}{\sigma_2}\right) \\
&\quad - K \cdot P(0,\tau) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2)}{\sigma_2}\right).
\end{aligned}
$$

Case (b): For $t_A \leq 0 < \tau < T < t_S$ the option price is given by

$$\begin{aligned} C(0) &= P(0,T) \cdot \Gamma(b,\alpha,\beta,\sigma,\tilde{\sigma},\tau,T,t_S) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2 - \frac{\sigma_2^2}{b})}{\sigma_2}\right) \\ &\quad - K \cdot P(0,\tau) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2)}{\sigma_2}\right). \end{aligned}$$

Case (c): For $t_A \leq 0 < \tau < t_S \leq T$ the option price is given by

$$\begin{aligned} C(0) &= P(0,T) \cdot \Gamma(b,\alpha,\beta,\sigma,\tilde{\sigma},\tau,T,t_S) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2 - \frac{\sigma_2^2}{b})}{\sigma_2}\right) \\ &\quad - K \cdot P(0,\tau) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2)}{\sigma_2}\right). \end{aligned}$$

Case (d): For $t_A \leq 0 < t_S \leq \tau < T$ the option price is given by

$$\begin{aligned} C(0) &= P(0,T) \cdot \Gamma(b,\alpha,\beta,\sigma,\tilde{\sigma},\tau,T,t_S) \cdot \Phi\left(\frac{y_0 - \left[\mu_2 - \rho\sigma_1\sigma_2 + \frac{\sigma_2^2}{b}(e^{-b(T-\tau)} - 1)\right]}{\sigma_2}\right) \\ &\quad - K \cdot P(0,\tau) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2)}{\sigma_2}\right). \end{aligned}$$

We end this section by remarking that the option-price dynamics presented in Case (a) of Proposition 3.6 coincides with a well-known bond-option formula that has been derived by several authors under the classical scenario in which no currency union is planned (see for example Björk, 2004, pp. 337, 338).

4 Simulation study

In this section we implement a Monte-Carlo simulation to assess the potentiality for option mispricing that might emerge from ignoring the specific exchange-rate and interest-rate dynamics during the run-up to a currency union. To this end, we assume that the currency union is announced at date t_A—implying that the option-price dynamics from Proposition 3.6 constitutes the 'correct' model—and simulate pricing paths of some zero-coupon bonds plus corresponding pricing paths of some bond options. We further suppose that, despite of the fact that the currency union has been announced, agents ignore the 'correct' option-price dynamics given by the Cases (b), (c), (d) of

Proposition 3.6 and erroneously presume instead that the bond-option dynamics from Proposition 3.6(a) still continues to be in force after t_A. As a result, agents misprice newly issued options by using this wrong option-price dynamics.

Our simulation starts in $t = 0$ and ends in $t = 2$. The dates relevant to the currency union are chosen as $t_A = 0.5$ and $t_S = 1.5$ implying an interim period of one year. For every parameter constellation we run a Monte-Carlo simulation with 10000 iterations and choose the distance between two points in time as 0.01. We set the mean-reversion level in the foreign short-rate process (6) to $c = 0.05$ and specify the irreversible exchange-rate fixing level as $s = \ln(1.00) = 0$. Following the line of argument in Wilfling (2003), we choose $\alpha = 2$ and, to simplify numerical procedures, set $\beta = 0$ implying that all $\Gamma(b, \alpha, \beta, \sigma, \tilde{\sigma}, \tau, T, t_S)$ function-values in Proposition 3.6 take on the constant value 1. For the parameters b, σ and $\tilde{\sigma}$ we choose the alternative setups shown in Table 2.

Based on these parameter constellations, we first simulate paths of the short rate $r(t)$ according to the dynamics given in Eq. (8). In a second step, we calculate five zero-coupon bond prices for the respective maturities of 1, 3, 6, 12, 24 months. Using these bond prices and the option-valuation formulas from Proposition 3.6, we then price six distinct options with strike prices K, option maturities τ and bond maturities T as shown in Table 1. It should be noted that these eleven bond and option prices represent arbitrage-free market prices.

Table 1 about here

In a next step, we price six newly issued zero-coupon bond options with strike prices $K \in \{0.915, 0.920, \ldots, 0.940\}$, option maturity $\tau = 2$ months and bond maturity $T =$ 14 months according to Proposition 3.6 at every of our supporting points in time. In contrast to our 'correct' pricing scheme, agents price these six options according to their erroneous assessment of option-price dynamics described above. In particular, using the 11 arbitrage-free bond and option prices observable in the market, agents calibrate their misspecified short-rate model consisting of the parameters b, c, σ thus obtaining different prices for the six newly issued options.

Table 2 displays the differences in the option prices obtained from (a) our pricing scheme (correct price), and (b) the misspecified valuation scheme employed by the agents (wrong price). We computed two measures of deviation, namely the average percentage deviation defined as the arithmetic mean of the values '$100 \times$ (wrong price $-$ correct price) $\div$ wrong price' and the average absolute percentage deviation defined as the mean of '$100 \times |\text{wrong price} - \text{correct price}| \div \text{wrong price}$'. Both measures were computed at the dates 3, 6, 9 months after the announcement date t_A.

Table 2 about here

Figure 1 about here

Table 2 reveals that both deviation measures exhibit (*ceteris paribus*) the tendency to increase as the strike price K increases. In particular, given the values of the parameters $b, \sigma, \tilde{\sigma}$ under the strike price $K = 0.940$, we observe substantial deviations of more than 61 per cent. To gain deeper insight into the nature of such deviations, Figure 1 plots the paths of average percentage deviations generated from the 10000 replications in our simulation study using the parameter values $b = 1, \sigma = 0.01, \tilde{\sigma} = 0.05$ and the distinct strike prices $K \in \{0.915, 0.920, 0.925, 0.930\}$. For comparative reasons, we have chosen a common range of the deviations along the vertical axis, thus truncating many deviation paths in the lower panels. In accordance with Case (a) of Proposition 3.6, all deviations are equal to zero before t_A and after t_S simply reflecting the fact that no mispricing occurs before the announcement of the currency union and after the union has been implemented. In all of the four panels, however, two striking features of the deviation dynamics during the interim period between t_A and t_S become apparent. (a) Deviations tend to exhibit a heteroskedastic fluctuation pattern over time. (b) During the first half of the interim period most deviations are positive, while we find more negative than positive deviations during the second half.

Figure 2 about here

To characterize the distribution of the pricing error, we have fitted kernel densities to the deviations measured at some specifically chosen points in time. Figure 2 displays the kernel densities obtained under the parameters $b = 1, \sigma = 0.01, \tilde{\sigma} = 0.05$ and strike prices $K = 0.915, 0.920$ at the dates $t_1 = 0.75$ (3 months after t_A), $t_2 = 1.0$ (6 months after t_A) and $t_3 = 1.25$ (9 months after t_A). Obviously, the kernel density at t_1 exhibits more mass at positive deviations while the reverse holds for the densities at t_2 and t_3. Moreover, higher strike prices appear to be associated with more leptokurtic error distributions.

5 Conclusions

Based on a continuous-time modeling framework characterizing the dynamic link between international interest rates in the run-up to a currency union, this paper derives closed-form valuation formulas for European call options on zero-coupon bonds. Taking into account the specific interest-rate dynamics induced by the switch in the exchange-rate regime, we extend the classical option-pricing framework and obtain novel pricing formulas. As the key result of our simulation study we find that disregarding the specific dynamic link between international interest rates prior to the currency union can generate substantial option-pricing errors.

It is obvious that our option-valuation formula may be used to price more complex contingent claims. As an example we could consider interest-rate floors which can be viewed as a portfolio of European call options on zero-coupon bonds. Interest-rate floors typically are among the most traded of all interest-rate derivatives so that our results should be of high value for traders in all sorts of financial and derivative markets located in the upcoming EMU accession countries. It is worth noting, however, that our option-price dynamics is not confined to the episode of a future entrance into a currency union. In fact, it is also applicable to comparable transitional periods in the international monetary system such as the run-up to an exchange-rate peg or the implementation of a currency board.

The exact forms of our option-pricing formulas crucially hinge on two of our specifications chosen in Section 2, namely (a) the Vasicek-dynamics of the foreign short rate $r^*(t)$ in Eq. (6), and (b) the driftless Brownian-motion specification of the exchange-rate fundamental $k(t)$ in Eq. (2). Clearly, alternative specifications are conceivable for both variables such as the classical short-rate models proposed by Cox et. al. (1985), Ho and Lee (1986) or Hull and White (1994) for $r^*(t)$.

In this context it should be recalled that the specification of the exchange-rate fundamental $k(t)$ is of particular importance since the k-dynamics characterizes the monetary policy regime during the run-up to the currency union. As described in Section 2, our (driftless) Brownian-motion specification represents a free-float exchange-rate regime between the countries involved. However, more interventionist exchange-rate policy stances prior to the currency union are conceivable and have indeed been pursued by some countries during the run-up to EMU (see Sondermann et al., 2010). Such active policy regimes can be modelled by Ornstein-Uhlenbeck and Brownian-bridge specifications for $k(t)$ (cf. Footnote 1) and one possible line of future research could be the investigation of how these alternative specifications affect our option-valuation dynamics derived in Proposition 3.6.

References

Björk, T., 2004. Arbitrage Theory in Continuous Time. Oxford University Press, New York.

Cox, J.C., Ingersoll, J.E., Ross, S.A., 1985. A Theory of the Term Structure of Interest Rates. Econometrica 53, 385-407.

De Grauwe, P., Dewachter, H., Veestraeten, D., 1999. Price Dynamics Under Stochastic Process Switching: Some Extensions and an Application to EMU. Journal of International Money and Finance 18 (2), 195-224.

Elliot, R.J., Kopp, P.E., 2005. Mathematics of Financial Markets, 2nd Edition. Springer Science+Business Media Inc., New York.

Ho, T.S.Y., Lee, S.-B., 1986. Term Structure Movements and Pricing Interest Rate Contingent Claims. Journal of Finance 41, 1011-1029.

Hull, J., White, A., 1994. Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models. Journal of Derivatives 2, 37-48.

Jamshidian, F., 1989. An Exact Bond Option Formula. Journal of Finance 44, 205-209.

Krugman, P., 1991. Target Zones and Exchange Rate Dynamics. Quarterly Journal of Economics 106, 669-682.

Sondermann, D., Trede, M, Wilfling, B., 2010. Estimating the Degree of Interventionist Policies in the Run-Up to EMU. Applied Economics, forthcoming.

Sutherland, A., 1995. State- and Time-Contingent Switches of Exchange Rate Regime. Journal of International Economics 38, 361-374.

Svensson, L.E.O., 1991a. Target Zones and Interest Rate Variability. Journal of International Economics 31, 27-54.

Svensson, L.E.O., 1991b. The Term Structure of Interest Rate Differentials in a Target Zone: Theory and Swedish Data. Journal of Monetary Economics 28, 87-116.

Vasicek, O., 1977. An Equilibrium Characterization of the Term Structure. Journal of Financial Economics 5, 177-188.

Wilfling, B., 2003. Interest Rate Volatility Prior to Monetary Union under Alternative Pre-Switch Regimes. German Economic Review 4, 433-457.

Wilfling, B., Maennig, W., 2001. Exchange Rate Dynamics in Anticipation of Time-Contingent Regime Switching: Modelling the Effects of a Possible Delay. Journal of International Money and Finance 20 (1), 91-113.

Appendix

To obtain the price dynamics of a European call option presented in Proposition 3.6 we have to calculate the expectation given on the right-hand side of Eq. (10):

$$E^Q\left[e^{-\int_0^\tau r(t)dt}\cdot\max\{P(\tau,T)-K,0\}|\phi_0\right]. \tag{A.1}$$

To this end, we follow the line of argument in Section 3.2 and consider the four distinct Cases (a) to (d) along with the corresponding bivariate probability distributions

$$\left(\int_0^\tau r(t)dt, r^*(\tau)\right), \qquad \text{[Case (a)]}$$

$$\left(\int_0^\tau r(t)dt, r^*(\tau)\left[1-e^{-b(T-\tau)}\right]+k(\tau)\left[be^{\frac{\tau-t_S}{\alpha}}-be^{\frac{T-t_S}{\alpha}}\right]\right), \qquad \text{[Case (b)]}$$

$$\left(\int_0^\tau r(t)dt, r^*(\tau)\left[1-e^{-b(T-\tau)}\right]+k(\tau)\left[be^{\frac{\tau-t_S}{\alpha}}-b\right]\right), \qquad \text{[Case (c)]}$$

$$\left(\int_0^\tau r(t)dt, r^*(\tau)\right). \qquad \text{[Case (d)]}$$

For ease of notation let us denote the first marginal distribution of any arbitrary bivariate random vector by X with expectation $\mu_1 \equiv E(X)$ and variance $\sigma_2^2 \equiv \mathrm{Var}(X)$ and the respective magnitudes of the second marginal distribution by $Y, \mu_2 \equiv E(Y)$ and $\sigma_2^2 \equiv \mathrm{Var}(Y)$. Furthermore, let us write the covariance of X and Y as $\mathrm{Cov}(X,Y) = \rho\sigma_1\sigma_2$. It is straightforward to obtain all these magnitudes for the case-specific bivariate random vectors from above by standard means of probability calculus.

Case (a): For $(X,Y)=\left(\int_0^\tau r(t)dt, r^*(\tau)\right)$ we have

$$\begin{aligned}
\mu_1 &= c\tau+\frac{r_0^*-c}{b}\left(1-e^{-b\tau}\right),\\
\sigma_1^2 &= \frac{\sigma^2}{2b^3}(2b\tau-3+4e^{-b\tau}-e^{-2b\tau}),\\
\mu_2 &= (r_0^*-c)e^{-b\tau}+c, \\
\sigma_2^2 &= \frac{\sigma^2}{2b}\left(1-e^{-2b\tau}\right),\\
\rho\sigma_1\sigma_2 &= \frac{\sigma^2}{2b^2}\left(1-e^{-b\tau}\right)^2.
\end{aligned} \tag{A.2}$$

Case (b): For $(X,Y)=\left(\int_0^\tau r(t)dt, r^*(\tau)\left[1-e^{-b(T-\tau)}\right]+k(\tau)\left[be^{\frac{\tau-t_S}{\alpha}}-be^{\frac{T-t_S}{\alpha}}\right]\right)$ we have

$$
\begin{aligned}
\mu_1 &= c\tau + \frac{r_0^* - c}{b}\left(1 - e^{-b\tau}\right) + (s - k_0)\left(e^{\frac{\tau - t_S}{\alpha}} - e^{\frac{-t_S}{\alpha}}\right), \\
\sigma_1^2 &= \frac{\sigma^2}{2b^3}\left(2b\tau - 3 + 4e^{-b\tau} - e^{-2b\tau}\right) + \frac{\tilde{\sigma}^2\alpha}{2} e^{\frac{2\tau - 2t_S}{\alpha}}\left(2\frac{\tau}{\alpha} - 3 + 4e^{-\frac{\tau}{\alpha}} - e^{-2\frac{\tau}{\alpha}}\right) \\
&\quad - 2\beta\frac{\sigma\tilde{\sigma}}{\alpha\sqrt{2b}} e^{-\frac{t_S}{\alpha}} \int_0^\tau \int_0^\tau e^{\frac{v}{\alpha}} e^{-bu} \min\left\{e^{2bu} - 1, v\right\} dv du, \\
\mu_2 &= \left(1 - e^{-b(T-\tau)}\right)\left[(r_0^* - c)e^{-b\tau} + c\right] + k_0\left[be^{\frac{\tau - t_S}{\alpha}} - be^{\frac{T - t_S}{\alpha}}\right], \qquad \text{(A.3)} \\
\sigma_2^2 &= \left(1 - e^{-b(T-\tau)}\right)^2 \frac{\sigma^2}{2b}\left(1 - e^{-2b\tau}\right) + \left(be^{\frac{\tau - t_S}{\alpha}} - be^{\frac{T - t_S}{\alpha}}\right)^2 \tilde{\sigma}^2\tau \\
&\quad + 2\beta\frac{\tilde{\sigma}\sigma}{\sqrt{2b}}\left(1 - e^{-b(T-\tau)}\right)\left(be^{\frac{\tau - t_S}{\alpha}} - be^{\frac{T - t_S}{\alpha}}\right) e^{-b\tau} \min\left\{e^{2b\tau} - 1, \tau\right\}, \\
\rho\sigma_1\sigma_2 &= \frac{\sigma^2}{2b^2}\left(1 - e^{-b\tau}\right)^2\left(1 - e^{-b(T-\tau)}\right) + b\alpha\tilde{\sigma}^2 e^{\frac{\tau - 2t_S}{\alpha}}\left(e^{\frac{T-\tau}{\alpha}} - 1 - \frac{\tau}{\alpha}e^{\frac{\tau}{\alpha}} + e^{\frac{\tau}{\alpha}} + \frac{\tau}{\alpha}e^{\frac{T}{\alpha}} - e^{\frac{T}{\alpha}}\right) \\
&\quad + \left(be^{\frac{\tau - t_S}{\alpha}} - be^{\frac{T - t_S}{\alpha}}\right)\frac{\tilde{\sigma}\sigma}{\sqrt{2b}}\beta \int_0^\tau e^{-bu} \min\left\{e^{2bu} - 1, \tau\right\} du \\
&\quad + \left(e^{-bT} - e^{-b\tau}\right)\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}\beta \int_0^\tau e^{\frac{u - t_S}{\alpha}} \min\left\{e^{2b\tau} - 1, u\right\} du.
\end{aligned}
$$

Case (c): For $(X, Y) = \left(\int_0^\tau r(t)dt, r^*(\tau)\left[1 - e^{-b(T-\tau)}\right] + k(\tau)\left[be^{\frac{\tau - t_S}{\alpha}} - b\right]\right)$ we have

$$
\begin{aligned}
\mu_1 &= c\tau + \frac{r_0^* - c}{b}\left(1 - e^{-b\tau}\right) + (s - k_0)\left(e^{\frac{\tau - t_S}{\alpha}} - e^{\frac{-t_S}{\alpha}}\right), \\
\sigma_1^2 &= \frac{\sigma^2}{2b^3}\left(2b\tau - 3 + 4e^{-b\tau} - e^{-2b\tau}\right) + \frac{\tilde{\sigma}^2\alpha}{2} e^{\frac{2\tau - 2t_S}{\alpha}}\left(2\frac{\tau}{\alpha} - 3 + 4e^{-\frac{\tau}{\alpha}} - e^{-2\frac{\tau}{\alpha}}\right) \\
&\quad - 2\beta\frac{\sigma\tilde{\sigma}}{\alpha\sqrt{2b}} e^{-\frac{t_S}{\alpha}} \int_0^\tau \int_0^\tau e^{\frac{v}{\alpha}} e^{-bu} \min\left\{e^{2bu} - 1, v\right\} du dv, \\
\mu_2 &= \left(1 - e^{-b(T-\tau)}\right)\left[(r_0^* - c)e^{-b\tau} + c\right] + k_0\left[be^{\frac{\tau - t_S}{\alpha}} - b\right], \qquad \text{(A.4)} \\
\sigma_2^2 &= \left(1 - e^{-b(T-\tau)}\right)^2 \frac{\sigma^2}{2b}\left(1 - e^{-2b\tau}\right) + \left(be^{\frac{\tau - t_S}{\alpha}} - b\right)^2 \tilde{\sigma}^2\tau \\
&\quad + 2\beta\frac{\tilde{\sigma}\sigma}{\sqrt{2b}}\left(1 - e^{-b(T-\tau)}\right)\left(be^{\frac{\tau - t_S}{\alpha}} - b\right) e^{-b\tau} \min\left\{e^{2b\tau} - 1, \tau\right\},
\end{aligned}
$$

$$
\begin{aligned}
\rho\sigma_1\sigma_2 &= \frac{\sigma^2}{2b^2}\left(1-e^{-b\tau}\right)^2\left(1-e^{-b(T-\tau)}\right) - b\alpha\tilde{\sigma}^2 e^{\frac{\tau-t_S}{\alpha}}\left(1-\frac{\tau}{\alpha}-e^{-\frac{\tau}{\alpha}}+\frac{\tau}{\alpha}e^{\frac{\tau-t_S}{\alpha}}+e^{-\frac{t_S}{\alpha}}-e^{\frac{\tau-t_S}{\alpha}}\right) \\
&\quad + \left(be^{\frac{\tau-t_S}{\alpha}}-b\right)\frac{\tilde{\sigma}\sigma}{\sqrt{2b}}\beta\int_0^{\tau} e^{-bu}\min\left\{e^{2bu}-1,\tau\right\}du \\
&\quad + \left(e^{-bT}-e^{-b\tau}\right)\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}\beta\int_0^{\tau} e^{\frac{u-t_S}{\alpha}}\min\left\{e^{2b\tau}-1,u\right\}du.
\end{aligned}
$$

Case (d): For $(X,Y)=\left(\int_0^\tau r(t)dt, r^*(\tau)\right)$ we have

$$
\begin{aligned}
\mu_1 &= c\tau+\frac{r_0^*-c}{b}\left(1-e^{-b\tau}\right)+(s-k_0)\left(1-e^{\frac{-t_S}{\alpha}}\right), \\
\sigma_1^2 &= \frac{\sigma^2}{2b^3}\left(2b\tau-3+4e^{-b\tau}-e^{-2b\tau}\right)+\frac{\tilde{\sigma}^2\alpha}{2}\left(2\frac{t_S}{\alpha}-3+4e^{-\frac{t_S}{\alpha}}-e^{-2\frac{t_S}{\alpha}}\right) \\
&\quad -2\beta\frac{\sigma\tilde{\sigma}}{\alpha\sqrt{2b}}e^{-\frac{t_S}{\alpha}}\int_0^{t_S}\int_0^{\tau} e^{\frac{v}{\alpha}}e^{-bu}\min\left\{e^{2bu}-1,v\right\}dudv,
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 &= (r_0^*-c)e^{-b\tau}+c, \qquad\qquad (A.5)\\
\sigma_2^2 &= \frac{\sigma^2}{2b}\left(1-e^{-2b\tau}\right), \\
\rho\sigma_1\sigma_2 &= \frac{\sigma^2}{2b^2}\left(1-e^{-b\tau}\right)^2-\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{-b\tau}\beta\int_0^{t_S} e^{\frac{u-t_S}{\alpha}}\min\left\{e^{2b\tau}-1,u\right\}du.
\end{aligned}
$$

Using this case-specific notation and noting that the bond price $P(\tau,T)$ is a function of the second marginal distribution Y, which justifies our expressing the bond price as $P(\tau,T,Y)$, we may rewrite the expectation (A.1) as

$$
\begin{aligned}
& E^Q\left[e^{-X}\cdot\max\{P(\tau,T,Y)-K,0\}|\phi_0\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x}\max\{P(\tau,T,y)-K,0\}f(x,y)dxdy, \qquad (A.6)
\end{aligned}
$$

where $f(x,y)$ represents the joint probability density function of (X,Y). The integrand of the double integral in Eq. (A.6) contains a maximum function. However, it is easy to check that the term $P(\tau,T,y)-K$ is monotone decreasing in y so that we can simplify the double integral by (a) calculating those case-specific values y_0 for which this term becomes zero, and (b) by changing the integration limits accordingly. These y_0-values are given as follows:

Case (a):

$$y_0 = \frac{-b\ln(K) - bc(T-\theta) + c\left(1-e^{-b(T-\theta)}\right) + \frac{\sigma^2}{4b^2}\left[2b(T-\theta) - 3 + 4e^{-b(T-\theta)} - e^{-2b(T-\theta)}\right]}{1-e^{-b(T-\theta)}}.$$

Case (b):

$$\begin{aligned} y_0 = & -b\ln(K) - bc(T-\theta) + c\left(1-e^{-b(T-\theta)}\right) + \frac{\sigma^2}{4b^2}\left[2b(T-\theta) - 3 + 4e^{-b(T-\theta)} - e^{-2b(T-\theta)}\right] \\ & + \frac{b\tilde{\sigma}^2\alpha}{4}e^{\frac{2T-2t_S}{\alpha}}\left(2\frac{T-\theta}{\alpha} - 3 - e^{-\frac{2T-2\theta}{\alpha}} + 4e^{-\frac{T-\theta}{\alpha}}\right) - bs\left(e^{\frac{T-t_S}{\alpha}} - e^{\frac{\theta-t_S}{\alpha}}\right) \\ & - b\beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{\frac{\theta-t_S}{\alpha}}\int_0^{T-\theta}\int_0^{T-\theta} e^{\frac{v}{\alpha}}e^{-bu}\min\left\{e^{2bu}-1, v\right\}dudv. \end{aligned}$$

Case (c):

$$\begin{aligned} y_0 = & -b\ln(K) - bc(T-\theta) + c\left(1-e^{-b(T-\theta)}\right) + \frac{\sigma^2}{4b^2}\left[2b(T-\theta) - 3 + 4e^{-b(T-\theta)} - e^{-2b(T-\theta)}\right] \\ & + \frac{b\tilde{\sigma}^2\alpha}{4}\left(2\frac{t_S-\theta}{\alpha} - 3 - e^{-\frac{2t_S-2\theta}{\alpha}} + 4e^{-\frac{t_S-\theta}{\alpha}}\right) - bs\left(1 - e^{\frac{\theta-t_S}{\alpha}}\right) \\ & - b\beta\frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}}e^{\frac{\theta-t_S}{\alpha}}\int_0^{t_S-\theta}\int_0^{T-\theta} e^{\frac{v}{\alpha}}e^{-bu}\min\left\{e^{2bu}-1, v\right\}dudv. \end{aligned}$$

Case (d):

$$y_0 = \frac{-b\ln(K) - bc(T-\theta) + c\left(1-e^{-b(T-\theta)}\right) + \frac{\sigma^2}{4b^2}\left[2b(T-\theta) - 3 + 4e^{-b(T-\theta)} - e^{-2b(T-\theta)}\right]}{1-e^{-b(T-\theta)}}.$$

Using these case-specific y_0-values, we can remove the maximum function and simplify the double integral in Eq. (A.6) to give

$$\int_{-\infty}^{y_0}\int_{-\infty}^{\infty} e^{-x}\left[P(\tau,T,y) - K\right]f(x,y)dxdy.$$

Inserting the explicit form of the bivariate normal probability density function for $f(x,y)$ and performing some straightforward manipulations, we obtain

$$\int_{-\infty}^{y_0}\left[P(\tau,T,y) - K\right]\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}\int_{-\infty}^{\infty} e^{-x}\frac{1}{\sqrt{2\pi}\sigma_1\sqrt{(1-\rho^2)}}e^{-\frac{1}{2}\frac{(x-(\mu_1+\rho y\frac{\sigma_1}{\sigma_2}-\rho\mu_2\frac{\sigma_1}{\sigma_2}))^2}{(1-\rho^2)\sigma_1^2}}dxdy.$$

The second integral in the latter term constitutes the expected value of a lognormal distribution yielding the equivalent expression

$$\int_{-\infty}^{y_0} [P(\tau, T, y) - K] \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} e^{-(\mu_1 + \rho y \frac{\sigma_1}{\sigma_2} - \rho\mu_2 \frac{\sigma_1}{\sigma_2}) + \frac{(1-\rho^2)\sigma_1^2}{2}} dy,$$

which can be expanded to give

$$\int_{-\infty}^{y_0} P(\tau, T, y) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} e^{-(\mu_1 + \rho y \frac{\sigma_1}{\sigma_2} - \rho\mu_2 \frac{\sigma_1}{\sigma_2}) + \frac{(1-\rho^2)\sigma_1^2}{2}} dy$$
$$- \int_{-\infty}^{y_0} K \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} e^{-(\mu_1 + \rho y \frac{\sigma_1}{\sigma_2} - \rho\mu_2 \frac{\sigma_1}{\sigma_2}) + \frac{(1-\rho^2)\sigma_1^2}{2}} dy.$$

The second integral in this last term can be expressed in terms of the standard normal cumulative distribution function (cdf) $\Phi(\cdot)$ yielding

$$\int_{-\infty}^{y_0} P(\tau, T, y) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} e^{-(\mu_1 + \rho y \frac{\sigma_1}{\sigma_2} - \rho\mu_2 \frac{\sigma_1}{\sigma_2}) + \frac{(1-\rho^2)\sigma_1^2}{2}} dy$$
$$- K \cdot P(0, \tau) \cdot \Phi\left(\frac{y_0 - (\mu_2 - \rho\sigma_1\sigma_2)}{\sigma_2}\right). \tag{A.7}$$

Note that $P(0, \tau)$ in Eq. (A.7) is the price of a zero-coupon bond maturing at time τ the (case-specific) form of which is established in Proposition 3.4.

Finally, substituting the bond price $P(\tau, T, y)$ by its explicit formulas for the four distinct Cases (a) to (d) and applying analogous steps as before, we are able to write the remaining integral in the expression (A.7) in terms of the standard normal cumulative distribution function. More precisely, the integral can be expressed as the product of the following three factors: (a) the bond price $P(0, T)$, (b) an auxiliary function $\Gamma(b, \alpha, \beta, \sigma, \tilde{\sigma}, \tau, T, t_S)$, and (c) a specific value of the cdf $\Phi(\cdot)$. The exact forms of the product are given in the respective first lines of the option-valuation formulas in Proposition 3.6 (Cases (a) to (d)). The precise form of the auxiliary function Γ, which can only be different from 1 for the Cases (b) to (d), are given as follows:

Case (b): $\Gamma(b, \alpha, \beta, \sigma, \tilde{\sigma}, \tau, T, t_S) =$

$$\exp\Bigg\{ - \beta e^{-\frac{t_S}{\alpha}} \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} \int_0^{T-\tau} \int_0^{T-\tau} e^{\frac{v+\tau}{\alpha}} e^{-b(u+\tau)} \min\left\{e^{2bu} - 1, v\right\} du dv$$
$$- \beta e^{-\frac{t_S}{\alpha}} \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} \int_0^{T} \int_0^{T} e^{\frac{v}{\alpha}} e^{-bu} \Big(\min\left\{e^{2bu} - 1, v, e^{2b\tau} - 1, \tau\right\}$$
$$- \min\left\{e^{2bu} - 1, v\right\} \Big) du dv \Bigg\}. \tag{A.8}$$

Case (c): $\Gamma(b,\alpha,\beta,\sigma,\tilde{\sigma},\tau,T,t_S) =$

$$\exp\Big\{ -\beta e^{-\frac{t_S}{\alpha}} \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} \int_0^{t_S-\tau} \int_0^{T-\tau} e^{\frac{v+\tau}{\alpha}} e^{-b(u+\tau)} \min\left\{e^{2bu}-1, v\right\} du dv$$

$$-\beta e^{-\frac{t_S}{\alpha}} \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} \int_0^{t_S} \int_0^{T} e^{\frac{v}{\alpha}} e^{-bu} \Big(\min\left\{e^{2bu}-1, v, e^{2b\tau}-1, \tau\right\}$$

$$- \min\left\{e^{2bu}-1, v\right\} \Big) du dv \Big\}. \tag{A.9}$$

Case (d): $\Gamma(b,\alpha,\beta,\sigma,\tilde{\sigma},\tau,T,t_S) =$

$$\exp\Big\{ \beta e^{-\frac{t_S}{\alpha}} \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} \int_0^{t_S} \int_\tau^{T} e^{\frac{v}{\alpha}} e^{-bu} \min\left\{e^{2bu}-1, v\right\} du dv$$

$$-\beta e^{-\frac{t_S}{\alpha}} \frac{\tilde{\sigma}\sigma}{\alpha\sqrt{2b}} \int_0^{t_S} \frac{1}{b} e^{\frac{v}{\alpha}} e^{-b\tau} \min\left\{e^{2b\tau}-1, v\right\} dv \Big\}. \tag{A.10}$$

In each of the three Cases (b) to (d) the function $\Gamma(b,\alpha,\beta,\sigma,\tilde{\sigma},\tau,T,t_S)$ depends *inter alia* on the parameter β which represents the (constant) correlation coefficient of the Wiener processes $W_1(t)$ and $\bar{W}(t)$ driving the foreign short rate $r^*(t)$ and the instantaneous short-rate differential $\mathrm{SRD}(t) = r(t) - r^*(t)$, respectively (cf. Section 2). If these Wiener processes are uncorrelated, i.e. for $\beta = 0$, the Γ-function takes on the value 1, which considerably simplifies our option-valuation formulas in Proposition 3.6.

Figures and Tables

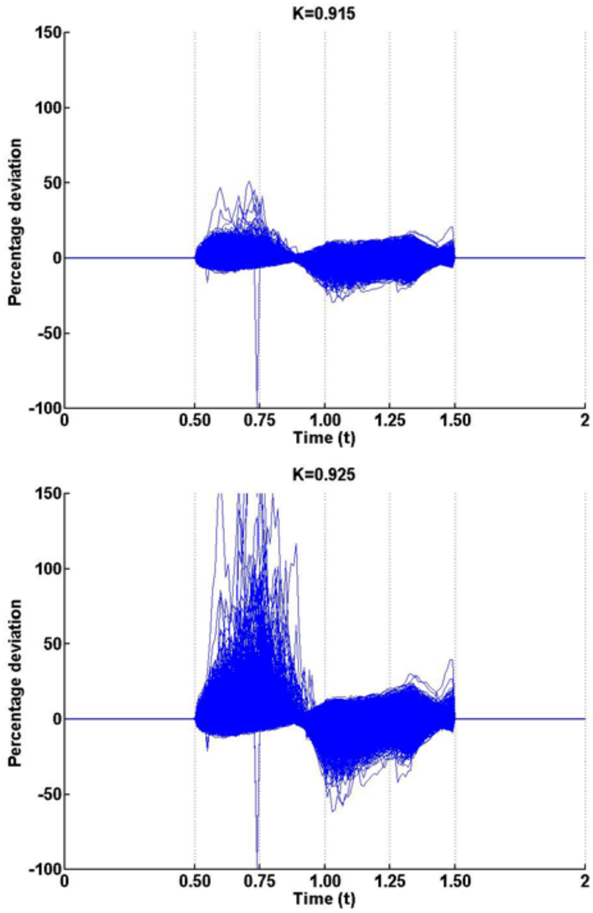

Figure 1: Average percentage deviations under the parameters $b = 1, \sigma = 0.01, \tilde{\sigma} = 0.05$ for alternative strike prices K

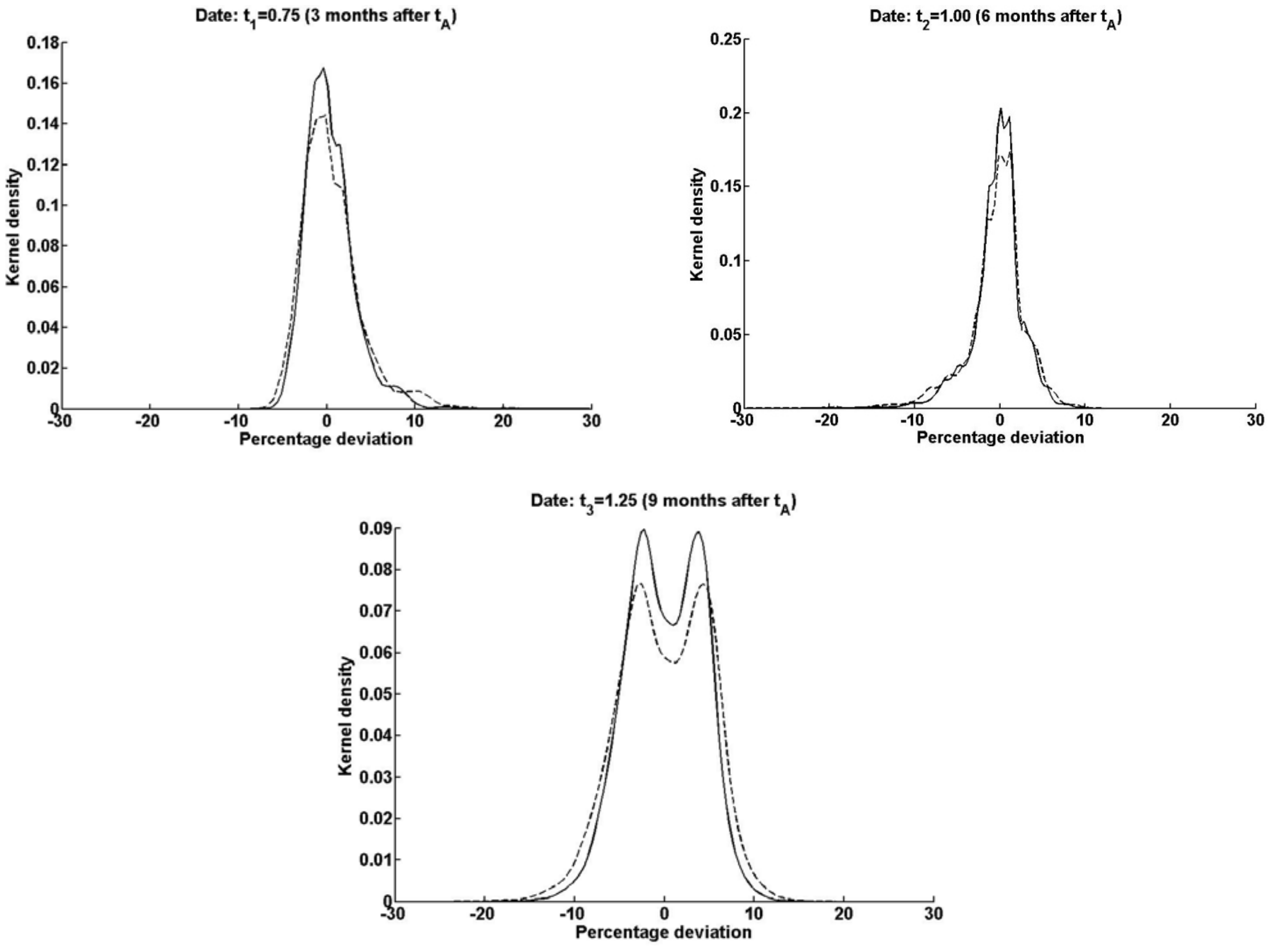

Figure 2: Kernel densities with $b = 1, \sigma = 0.01, \tilde{\sigma} = 0.05, K = 0.915$ (solid lines), $K = 0.92$ (dashed lines) for alternative points in time after t_A

Table 1

Parameters of options used for yield inversion

τ (in months)	T (in months)	K
1	3	0.99
1	6	0.98
1	12	0.96
3	6	0.99
3	12	0.97
6	12	0.98

Table 2

Deviations of option prices for alternative parameters when valuing (a) under the 'correct' and (b) under the 'wrong' bond-option dynamics

Time after t_S (in months)	Parameter setup	$K = 0.915$ Average perc. dev.	$K = 0.915$ Average abs. perc. dev.	$K = 0.920$ Average perc. dev.	$K = 0.920$ Average abs. perc. dev.
3	$b = 1$,	0.413	1.928	0.720	2.442
6	$\sigma = 0.015$,	-0.262	2.025	-0.383	2.384
9	$\tilde{\sigma} = 0.050$	-0.037	3.584	-0.073	4.208
3	$b = 1$,	0.072	0.950	0.109	1.119
6	$\sigma = 0.015$,	-0.087	0.962	-0.113	1.118
9	$\tilde{\sigma} = 0.025$	0.146	2.149	0.169	2.519
3	$b = 1$,	0.374	2.094	0.618	2.569
6	$\sigma = 0.010$,	-0.309	2.055	-0.453	2.432
9	$\tilde{\sigma} = 0.050$	0.035	3.489	0.007	4.077
3	$b = 1$,	0.086	1.005	0.122	1.176
6	$\sigma = 0.010$,	-0.130	0.916	-0.170	1.070
9	$\tilde{\sigma} = 0.025$	0.139	2.091	0.155	2.439
3	$b = 2$,	0.596	2.319	0.839	2.793
6	$\sigma = 0.015$,	-0.339	1.464	-0.474	1.747
9	$\tilde{\sigma} = 0.050$	-0.159	3.049	-0.219	3.558
3	$b = 2$,	0.078	1.124	0.114	1.311
6	$\sigma = 0.015$,	-0.047	0.544	-0.068	0.636
9	$\tilde{\sigma} = 0.025$	0.027	1.529	0.024	1.781

Note: The deviation measures 'Average percentage deviation' and 'Average absolute percentage deviation' are defined as the arithmetic means of the values '$100 \times (\text{wrong price} - \text{correct price}) \div \text{wrong price}$' and '$100 \times |\text{wrong price} - \text{correct price}| \div \text{wrong price}$', respectively.

Table 2 (continued)

Time after t_S (in months)	Parameter setup	$K = 0.925$ Average perc. dev.	$K = 0.925$ Average abs. perc. dev.	$K = 0.930$ Average perc. dev.	$K = 0.930$ Average abs. perc. dev.
3	$b = 1$,	1.488	3.487	3.697	6.102
6	$\sigma = 0.015$,	−0.515	2.861	−0.434	3.654
9	$\tilde{\sigma} = 0.050$	−0.142	5.110	−0.295	6.562
3	$b = 1$,	0.197	1.388	0.475	1.933
6	$\sigma = 0.015$,	−0.120	1.335	0.043	1.785
9	$\tilde{\sigma} = 0.025$	0.203	3.050	0.257	3.905
3	$b = 1$,	1.200	3.455	2.833	5.509
6	$\sigma = 0.010$,	−0.681	2.966	−0.948	3.701
9	$\tilde{\sigma} = 0.050$	−0.051	4.907	−0.179	6.173
3	$b = 1$,	0.191	1.429	0.359	1.861
6	$\sigma = 0.010$,	−0.226	1.282	−0.288	1.591
9	$\tilde{\sigma} = 0.025$	0.174	2.928	0.194	3.669
3	$b = 2$,	1.342	3.598	2.634	5.307
6	$\sigma = 0.015$,	−0.710	2.177	−1.151	2.881
9	$\tilde{\sigma} = 0.050$	−0.319	4.271	−0.507	5.341
3	$b = 2$,	0.176	1.576	0.303	1.991
6	$\sigma = 0.015$,	−0.102	0.767	−0.169	0.968
9	$\tilde{\sigma} = 0.025$	0.017	2.131	−0.001	2.653

Note: The deviation measures 'Average percentage deviation' and 'Average absolute percentage deviation' are defined as the arithmetic means of the values '$100 \times (\text{wrong price} - \text{correct price}) \div \text{wrong price}$' and '$100 \times |\text{wrong price} - \text{correct price}| \div \text{wrong price}$', respectively.

Table 2 (continued)

Time after t_S (in months)	Parameter setup	$K = 0.935$ Average perc. dev.	Average abs. perc. dev.	$K = 0.940$ Average perc. dev.	Average abs. perc. dev.
3	$b = 1$,	11.555	14.751	56.106	61.371
6	$\sigma = 0.015$,	0.977	5.827	10.314	16.466
9	$\tilde{\sigma} = 0.050$	-0.702	9.461	-1.354	17.122
3	$b = 1$,	1.667	3.613	9.590	12.978
6	$\sigma = 0.015$,	1.232	3.298	9.891	12.432
9	$\tilde{\sigma} = 0.025$	0.297	5.741	0.571	11.791
3	$b = 1$,	8.332	11.645	34.768	39.374
6	$\sigma = 0.010$,	-0.636	4.871	4.693	10.522
9	$\tilde{\sigma} = 0.050$	-0.512	8.388	-1.506	13.541
3	$b = 1$,	0.973	2.887	4.985	7.726
6	$\sigma = 0.010$,	-0.165	2.186	2.634	5.110
9	$\tilde{\sigma} = 0.025$	0.206	4.957	0.234	8.139
3	$b = 2$,	6.956	10.234	30.862	35.236
6	$\sigma = 0.015$,	-1.979	4.092	-3.388	6.196
9	$\tilde{\sigma} = 0.050$	-0.916	7.144	-1.917	10.739
3	$b = 2$,	0.666	2.788	2.262	5.140
6	$\sigma = 0.015$,	-0.313	1.316	-0.518	1.946
9	$\tilde{\sigma} = 0.025$	-0.051	3.513	-0.191	5.210

Note: The deviation measures 'Average percentage deviation' and 'Average absolute percentage deviation' are defined as the arithmetic means of the values '$100 \times (\text{wrong price} - \text{correct price}) \div \text{wrong price}$' and '$100 \times |\text{wrong price} - \text{correct price}| \div \text{wrong price}$', respectively.