The Valuation of Currency Options

Nahum Biger and John Hull

Both Nahum Biger and John Hull are Associate Professors of Finance in the Faculty of Administrative Studies, York University, Canada.

Introduction

Since Black and Scholes [1] published their path-breaking paper, option pricing theory has received considerable attention in the literature. Many authors have shown how the basic Black-Scholes model can be extended with its underlying assumptions being relaxed. The model has also found many applications in finance. Smith [6] provides a good overall review of the subject.

Options on a foreign currency can be defined in the same way as options on a stock. For example, a European call option on a foreign currency is an option to buy one unit of the currency on a predetermined date at a predetermined exchange rate. At the time of writing there is no well-organized market for foreign currency options. However, the Philadelphia, Montreal and Vancouver stock exchanges have all submitted proposals for the creation of such a market.

The prices of foreign currency options can be important in determining the values of other financial contracts. Feiger and Jacquillat [2] consider one such contract, the currency option bond. (This is a bond where the holder can choose the currency in which coupons and principal are paid according to a pre-determined exchange rate.) Feiger and Jacquillat point out that a two-currency, currency option bond is equivalent to a single-currency bond plus a foreign currency option. Thus:

$$P = B + cp$$

where P is the price of a bond paying either \$1 or £p at time T, B is the price of a pure discount bond paying \$1 at time T and c is the price of a European call option to purchase £1 for a dollar price of $^1/_p$ at time T.

Options would add completeness to the foreign exchange market. For example, by combining a long (short) position in a currency with a put (call) option, a corporation or private investor could limit downside risk while benefiting from favorable exchange movements. This partial hedge would be an alternative to the total hedge that can be obtained using forward markets. Feiger and Jacquillat [2] point out that foreign currency options might also be attractive to a corporation that is uncertain whether it will have a long position in a currency (*e.g.*, because it is bidding for a foreign currency denominated contract). They show that the combination of a forward contract to sell the currency at time T and a call option to buy it at time T provides a hedge not available with forward contracts alone.

Foreign currency options have received relatively little attention in the literature. Feiger and Jacquillat [2] develop a valuation model for two-currency, currency option bonds, assuming a joint stochastic process for the exchange rate, home interest rate and foreign interest rate. More recently Stulz [9] has developed a series of analytical formulas for European put and call options on the minimum or maximum of two risky assets. These can be applied to value currency options.

This paper first provides a direct derivation of valuation formulas for European put and call foreign exchange options using the Black-Scholes methodology. It then shows that the same formulas can be derived by assuming the expectations theory of exchange rates and the capital asset pricing model. The formulas are illustrated with an example. The paper assumes a stochastic process for just one variable. This produces far simpler valuation formulas than those of Feiger and Jacquillat [2]. The approach also has the advantage that it provides useful insights into the key role played by the forward exchange rate in the valuation process.

Valuation Using Black-Scholes Methodology

In this section we value European put and call options on a foreign currency under the following assumptions:

a) the price of one unit of foreign currency follows a Geometric Brownian Motion;

b) the foreign exchange market operates continuously with no transaction costs or taxes;

c) the risk-free interest rates in both the foreign country and the home country are constant during the life of the option.

In the Appendix we show how the assumptions in (c) can be relaxed.

When valuing options on a stock, Black and Scholes [1] make the following assumptions: (i) the stock follows a Geometric Brownian Motion, (ii) there are no penalties for short sales, (iii) transaction costs and taxes are zero, (iv) the market operates continuously, (v) the risk-free interest rate is constant, and (vi) the stock pays no dividend. They show that it is possible to create a riskless hedge using only the stock and European call options written on the stock. The holdings in the hedge must be continuously adjusted so that the ratio of the number of stock held to call options sold is always $\partial c/\partial S$ where S is the stock price and c is the call price. In equilibrium the return on the hedge must be the risk-free interest rate and this leads to a differential equation relating c and S. The solution to the differential equation is the well-known Black-Scholes option pricing formula:

$$c = S \cdot N\left\{\frac{\ln(S/X) + [r + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} - e^{-rT}X \cdot N\left\{\frac{\ln(S/X) + [r - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \quad (1)$$

where X is the exercise price, T is the exercise date, σ^2 is the instantaneous variance of the stock's return, r is the risk-free interest rate and N is the cumulative standard normal distribution function.

As Black and Scholes assume that no dividends are paid on the stock during the life of the option, their model cannot be directly applied to value an option on a foreign currency. This is because an investor who wishes to hold a foreign currency should always choose short-term risk-free foreign currency bonds in preference to holding the foreign currency in some non-interest-bearing account. A holding of a foreign currency can, therefore, be considered as giving a return equal to the foreign risk-free rate and valuing an option on a foreign currency is, therefore, essentially the same problem as valuing an option on a stock paying a continuous dividend. Merton [5] and Smith [6] consider the latter problem on the assumption that the dividend yield, δ, is constant. They show that a riskless hedge can be constructed as above, with the same hedge ratio, and that the valuation formula becomes:

$$c = e^{-\delta T}S \cdot N\left\{\frac{\ln(S/X) + [r - \delta + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} - e^{-rT}X \cdot N\left\{\frac{\ln(S/X) + [r - \delta - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \quad (2)$$

If the risk-free interest rate that can be earned on the foreign currency holding, r^*, is assumed to be constant, the "dividend yield" from an investment in the foreign currency (when measured in terms of the home currency) is constant and equal to r^*. Hence, if we redefine variables as follows:

S: spot price of one unit of the foreign currency;
σ^2: instantaneous variance of the return on a foreign currency holding;
X, T: exercise price and date of a European call option to purchase one unit of the foreign currency;
r: risk-free rate of interest in the home country;

then, under the assumptions given at the beginning of this section, Equation (2) provides a valuation formula for a European call option written on the foreign currency when $\delta = r^*$:

$$c = e^{-r^*T} S \cdot N\left\{\frac{\ln(S/X) + [r - r^* + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} - e^{-rT} X \cdot N\left\{\frac{\ln(S/X) + [r - r^* - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \quad (3)$$

Define F as the forward rate on the foreign currency for a contract with delivery date T. Interest rate parity theory implies:

$$\ln(F/S) = (r - r^*)T \quad (4)$$

and substituting (4) into (3) the valuation formula reduces to:

$$c = e^{-rT} F \cdot N\left\{\frac{\ln(F/X) + (\sigma^2/2)T}{\sigma\sqrt{T}}\right\} - e^{-rT} X \cdot N\left\{\frac{\ln(F/X) - (\sigma^2/2)T}{\sigma\sqrt{T}}\right\} \quad (5)$$

To value a European put option with exercise price X and exercise date T, an argument similar to that in Merton [5] relating put and call prices for a stock can be used. The terminal value of the put option is the same as the terminal value of a portfolio consisting of:

i) a call option with exercise price X and exercise date T;
ii) $(X - F)e^{-rT}$ of risk-free bonds;
iii) a forward contract with delivery date T to sell 1 unit of the foreign currency for price F.

Since the value of the forward contract is zero, it follows that the value of the put option, p, is given by:

$$p = c + (X - F)e^{-rT}. \quad (6)$$

Since $1 - N(q) = N(-q)$ we obtain:

$$p = e^{-rT} X \cdot N\left\{\frac{\ln(X/F) + (\sigma^2/2)T}{\sigma\sqrt{T}}\right\} - e^{-rT} F \cdot N\left\{\frac{\ln(X/F) - (\sigma^2/2)T}{\sigma\sqrt{T}}\right\}. \quad (7)$$

It is clear from (5) and (7) that the forward rate plays a central role in the valuation of foreign currency options. c and p depend on F, X, σ, T and r rather than on S, X, σ, T and r (which are the parameters involved in valuing stock options). This suggests an insightful extension of the Black-Scholes approach where an investor forms a riskless hedge by combining forward contracts with a short position in call options. The details are presented in the Appendix. Formulas similar to (5) and (7) are produced without the assumption of the foreign risk-free rate being constant.

Alternative Approach

It is interesting that the formulas in (5) and (7) can also be derived under the following assumptions:

a) the covariance between the foreign exchange rate and the returns from an international market portfolio is zero;[1]
b) the spot rate follows a Geometric Brownian Motion with instantaneous standard deviation, σ;
c) the international Sharpe-Lintner capital asset pricing model holds.[2]

Assumption (a) implies that the covariance of the returns from an option on the foreign currency and the returns from the international market portfolio is zero. From assumption (c) it follows that options on the foreign currency should be valued at their expected terminal value, discounted at the risk-free rate, *i.e.*,

$$c = e^{-rT}\int_X^\infty (S_T - X)\, f(S_T \mid S_O)\, dS_T$$

and

$$p = e^{-rT}\int_O^X (X - S_T)\, f(S_T \mid S_O)\, dS_T$$

where S_T is the price of the foreign currency at time T, S_O is the price of the foreign currency at time O and f is the probability distribution of S_T conditional on S_O.

As is well-known, assumption (b) implies that $\ln(S_T/S_O)$ is normally distributed with standard deviation $\sigma\sqrt{T}$. The lognormality property of Geometric Brownian Motion has been found appealing in the case of stocks by many researchers. In the case of a foreign currency it is worth noting that it has an added appeal. If the price of the foreign currency expressed in terms of the home currency ($= S_T$) is lognormal then the

[1]This is at best only approximately true. Recent theoretical and empirical work suggests that the covariance may be non-zero. See for example Hansen and Hodrick [4] and Stulz [8].

[2]See, for example, Grauer et al. [3] for a discussion of the international capital asset pricing model.

price of the home currency expressed in terms of the foreign currency ($= 1/S_T$) is also lognormal.

Using integrals of the lognormal distribution that are derived in an appendix to Sprenkle [7] and reproduced by Smith [6] it follows that:

$$c = e^{\rho T - rT} S \cdot N \left\{ \frac{\ln(S/X) + [\rho + \sigma^2/2]T}{\sigma\sqrt{T}} \right\} - e^{-rT} X \cdot N \left\{ \frac{\ln(S/X) + [\rho - \sigma^2/2]T}{\sigma\sqrt{T}} \right\} \quad (8)$$

where ρ is the expected average growth in the price of the foreign currency.

From assumptions (a) and (c) it follows that the forward rate, F, is an unbiased predictor of the spot rate at time T. Hence,

$$\rho T = \ln F/S \quad (9)$$

When (9) is substituted into (8) we obtain precisely the same formula for c as (5). A similar analysis leads to the same formula for p as (7).

Example

To illustrate these results we consider a U.S. company that is due to receive Can. $1M in 3 months' time. We assume that the 3-month forward rate for the Canadian dollar is $0.80, that the 3-month risk-free interest rate is 2.5%, and that the standard deviation of $\ln(S_T/S_O)$ is 0.02 where, when S_O is the spot rate at a certain time, S_T is the spot rate 3 months later.

The price, p, of a put option to sell Can. $1 for $0.80 in 3 months is given by substituting $X = 0.80$, $F = 0.80$, $e^{-rT} = \frac{1}{1.025}$ and $\sigma\sqrt{T} = 0.02$ in equation (7):

$$p = \frac{1}{1.025} \times 0.80 \times N(0.01) - \frac{1}{1.025} \times 0.80 \times N(-0.01) = 0.0062$$

Similarly, the price of a put option to sell Can. $1 for $0.79 in 3 months is $0.0025 and the price of a put option to sell Can. $1 for $0.81 in 3 months is $0.0123.

Thus, given an efficiently-functioning market for currency options, the company could choose between a number of different hedging strategies. At a cost of $6,200 it could ensure that it would obtain a *minimum* of $0.80M for the Can. $1M; at a cost of $2,500 it could ensure that it would obtain a minimum of $0.79M; at a cost of $12,300 it could ensure that it would obtain a minimum of $0.81M. All of these strategies would be alternatives to using the forward markets where, at virtually no cost, the company could ensure that it would obtain *exactly* $0.80M.

Call options could be used in a similar way by a company due to pay out a certain sum of money denominated in a foreign currency at a certain time in the future.

Summary

The results in this paper provide an elegant application of the Black-Scholes methodology. It is interesting that the forward rate plays a central role in the valuation formulas. The generality of the analysis in the Appendix suggests that the forward rate is central to the valuation of European put and call options on any income-producing security.

References

1. F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* (1973), pp. 637–659.
2. G. Feiger and B. Jacquillat, "Currency Option Bonds, Puts, and Calls on Spot Exchange and the Hedging of Contingent Foreign Earnings," *Journal of Finance* (1979), pp. 1129–1139.
3. F. L. Grauer, R. H. Litzenberger and R. E. Stehle, "Sharing Rules and Equilibrium in an International Capital Market Under Uncertainty," *Journal of Financial Economics* (1976), pp. 233–256.
4. L. Hansen and R. Hodrick, "Forward Exchange Rates as Optimal Predictors of Future Spot Rates: An Econometric Analysis," *Journal of Political Economy* (1980), pp. 829–853.
5. R. C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* (1973), pp. 141–183.
6. C. W. Smith Jr., "Option Pricing: A Review," *Journal of Financial Economics* (1976), pp. 3–51.
7. C. M. Sprenkle, "Warrant Prices as Indicators of Expectations and Preferences," in P. Cootrev, ed., *The Random Character of Stock Market Prices*, Cambridge, Mass, MIT Press (1964), pp. 412–474.
8. R. M. Stulz, "A Model of International Asset Pricing," *Journal of Financial Economics* (1981), pp. 383–406.
9. R. M. Stulz, "Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications," *Journal of Financial Economics* (1982), pp. 161–185.
10. S. M. Turnbull, "A Note on the Pricing of Foreign Currency Options," Working Paper, Department of Economics, University of Toronto, (March 1983).

Appendix

An investor can form a riskless hedge by combining a long position in forward contracts with a short position in call options. At any given time, t, he must adjust his portfolio so that the ratio of forward contracts held to call options sold is $\partial c/\partial F_t$ where $F_t = Fe^{-r(T-t)}$, and F is the forward rate at time t for a contract with delivery date T. (This hedge ratio is explained by the fact that when F increases by δF the value of a forward contract increases by $\delta Fe^{-r(T-t)}$.) If it is assumed that

a) F follows a Geometric Brownian Motion;

b) the forward exchange market operates continuously with no transaction costs and no taxes;

c) the home risk-free rate, r, is constant,

an analysis analogous to that of Black and Scholes [1] provides a differential equation relating c and F, the solution of which is

$$c = F_t \cdot N \left\{ \frac{\ln(F_t/X) + [r + (\sigma_F^2/2)]T}{\sigma_F\sqrt{T}} \right\} - e^{-rT} X \cdot N \left\{ \frac{\ln(F_t/X) + [r - (\sigma_F^2/2)]T}{\sigma_F\sqrt{T}} \right\} \quad \text{(A-1)}$$

where σ_F is the instantaneous standard deviation of F. (This is the same as the standard deviation of F_t under Geometric Brownian Motion.)

Since $F_t = Fe^{-rT}$ at $t = 0$, (A-1) becomes:

$$c = e^{-rT} F \cdot N \left\{ \frac{\ln(F/X) + (\sigma_F^2/2)T}{\sigma_F\sqrt{T}} \right\} - e^{-rT} X \cdot N \left\{ \frac{\ln(F/X) - (\sigma_F^2/2)T}{\sigma_F\sqrt{T}} \right\} \quad \text{(A-2)}$$

Thus, if it can be assumed that the forward rate follows a Geometric Brownian Motion it is not necessary to assume a constant foreign risk-free rate. Equation (A-2) is the same valuation formula as (5) with σ being replaced by σ_F. When we make the additional assumption that the foreign risk-free interest rate, r^*, is constant, F follows a Geometric Brownian Motion if and only if S does and $\sigma_F = \sigma$. The result in (A-2) then becomes equivalent to the one in (5).

In a recent working paper, Turnbull [10] has extended the results in this paper still further to deal with situations in which interest rates in both the domestic and foreign country are stochastic.