Westfälische Wilhelms-
Universität Münster

Institut für Kreditwesen
Direktor: Prof. Dr. Andreas Pfingsten

Institut für Kreditwesen

Diskussionsbeitrag 99-04

Three Dimensions of Shortfall Risk:

Transformation and Extension of Sen's Poverty Index

Dipl.-Kfm. Frank Eggers

Prof. Dr. Andreas Pfingsten

Dipl.-Kfm. Sven Rieso

ISSN 0949-6610

September 2000

THREE DIMENSIONS OF SHORTFALL RISK: TRANSFORMATION AND EXTENSION OF SEN'S POVERTY INDEX [‡‡]

Frank Eggers
Andreas Pfingsten [*]
Sven Rieso

Abstract

Incidence, intensity, and inequality of possible shortfalls relative to a target are features of shortfall risk. They are all integrated in a single measure by transferring an index known from the theory of poverty measurement, the Sen-Index. The index is reformulated to be used as a downside-risk measure for yield distributions in general. The resulting new Sen-Index provides a complete ranking of yield distributions, incorporating, unlike the presently very popular Value at Risk, the three aggregated dimensions of shortfall risk named above. Additionally, displaying the individual components separately makes the structure of a distribution's shortfall risk more transparent. A discussion of selected axioms highlights a specific weakness of the new Sen-Index with respect to a certain degree of sensitivity to inequality in the distribution of downside yields. To overcome the detected shortcoming, a further modification is presented.

[‡‡] The authors are indebted to the participants of the 3rd International Stockholm Seminar on Risk Behaviour and Risk Management, June 14-16, 1999, the Symposium on Operations Research in Magdeburg (SOR '99), September 1-3, 1999, the 6th Annual Meeting of the Deutsche Gesellschaft fuer Finanzwirtschaft (DGF), September 24, 1999, and the 8th Symposium on Finance, Banking and Insurance in Karlsruhe, December 15-17, 1999 for helpful discussions and comments on this and related work.

[*] Institut für Kreditwesen, Westfälische Wilhelms-Universität Münster, Universitätsstraße 14-16, D-48143 Münster, Germany; Phone: +49 (0)251/83-22881, email: 21anpf@wiwi.uni-muenster.de; www: http://www.bank-bwl.de

Contents

1 Introduction

Risk management is an important topic in portfolio management. It is fair to say that for this purpose μ-σ-approaches have been most prominent for quite some time, implying that risk is identified more or less with the standard deviation of yields. More recently this has been challenged by the increasing use of the *Value at Risk* (VaR). Its application has been accepted, by regulatory authorities for measuring banks' market risks, and was in fact thereby pushed.

In a nutshell, the VaR is the loss (in money units) which is exceeded within a specified period of time with a given probability. As a monetary figure it is easy to interpret and to communicate, in particular on the senior management level, and it can be added across business units. Hence it should not be surprising how popular this new measure is.

In technical terms one should note that the VaR only takes one single point of the whole distribution function into account. This feature is clearly unsatisfactory and, together with others, stimulated a number of critical papers (cf. Artzner et al. (1997), Guthoff et al. (1997), Johanning (1998)). Yet only a few other indices have been suggested since.

From an unbiased perspective one has to admit that the risk of not meeting a certain target, the shortfall risk, has at least three characteristics:[1]

- Incidence: frequency of shortfalls.
- Intensity: severity of shortfalls.
- Inequality: distribution of shortfalls.

The VaR requires a given incidence and measures the intensity in a very special way. This is obviously not adequate.

Suppose three or more alternative portfolios with equal expected yield are compared with respect to their risk. Then it might be tempting to describe risk by all three characteristics. A simple application of Arrow's (1963) famous impossibility theorem however should convince the reader that an uncontroversial risk-ranking is then not achievable in general. Needed is an index which, based on more or less implicit assumptions, provides a complete ranking.

When comparing portfolios A and B, still, it would be nice to know not only *that* but also *why* A is riskier than B, say. More specifically, one would probably like to see the differences with respect to the three characteristics mentioned in order to be able to say, for example, that the higher overall risk of A is due to a much higher frequency of shortfalls which are about as intense as those of B and slightly less dispersed.

The main purpose of our paper is to contribute a shortfall-risk measure that is capable of supplying both, a complete ranking and detailed information on the single characteristics of shortfall risk. We are happy to draw upon the work of Nobel laureate Amartya Sen. His ingenious work in welfare economics has produced, among others, a poverty measure later called *Sen-Index*. It turns out that a modification of this index may nicely serve our purpose.[2]

[1] For the context of poverty measurement, a graphical decomposition of these dimensions was presented by Jenkins and Lambert (1997).

[2] Some core ideas of this work have been published in German language, c.f. Eggers et al. (1999).

At first, some readers might feel uneasy with a transfer of concepts from welfare economics to finance. But this transfer has some tradition. In the early seventies a number of papers have examined and exploited analogies between inequality measurement and risk measurement (cf. Atkinson (1970), Dasgupta et al. (1973), Rothschild and Stiglitz (1970 and 1973), and the surveys of Eichhorn and Vogt (1990) and Nermuth (1993)). Income inequality was calculated for a whole distribution and so was risk in terms of "dispersion". In the present context, focusing on shortfall risk, we are only interested in the lower part of the distribution of yields. And this is indeed very closely related to poverty measurement where only the incomes of the poor, i.e. also the lower percentiles of a distribution function, matter.

The paper is organized as follows: Section 2 gives a brief summary of the basic definitions from the theory of poverty measurement as far as they are needed for the interpretation of the Sen-Index in the original context. The Sen-Index itself is also presented. In Section 3 problems arising from the transfer of the original index into the risk management framework are addressed. By adapting the index to the new application without changing its nature, the new Sen-Index is developed. Section 4 briefly caracterizes the new measure with respect to selected axioms. Section 5 then presents a further modification of the new Sen-Index to overcome a particular defect detected in Section 4. Section 6 concludes the paper.

2 Basic Definitions and Concepts Borrowed from the Theory of Poverty Measurement

In the measurement of poverty two distinct operations have to be performed:[3] The identification of the poor and the aggregation of the characteristics among the different poor individuals into an aggregate measure. Both issues are widely debated in the literature. The problems connected with the identification of the individuals' minimum needs and the income level marking off the set of poor people are important in poverty measurement. The poor are defined as the set of people whose income does not exceed the so-called *poverty line* z. For our analysis, we assume that the corresponding figure in shortfall-risk measurement, the *target yield* z, is an externally set limit, which may for example be determined indirectly by the amount of equity a bank has assigned to a trading unit. The issue of aggregation thus is the very field of poverty measurement which we want to exploit.

Poverty is measured by *poverty indices*. From the vast variety of indices only some very basic ones are needed for the construction of the Sen-Index which is to be transferred later.

The *head-count ratio* expresses the proportion of the total population that has been identified as being poor, i.e. whose income $y < z$.[4] The head-count measure shows the *incidence* of poverty:

$$H = q/n, \tag{1}$$

where n is the total number of individuals and q the number of poor people.

For y_i denoting the income of person i, the amount by which the income of person i falls

[3] Cf. Sen (1979), p. 285.
[4] Cf. Sen (1979), p. 294.

short of the poverty line z, $g_i = z - y_i$, is the *income gap* of individual i. The sum of all income gaps g_i is called *poverty gap*. The *income-gap ratio* is defined as the percentage shortfall from the poverty line of the average income of a poor person. This index captures the average *intensity* of poverty among the poor:

$$I = \sum_{i \in Q(z)} \frac{g_i}{qz}, \tag{2}$$

where $Q(z)$ is the set of poor people.

The *Gini-Index* can be used as a measure of the *inequality* of the income distribution among the poor. When incomes are arranged in increasing order, i.e. $y_i \leq y_{i+1}$ for $i = 1, ..., q-1$, it is defined as follows :

$$G = \frac{1}{2q^2 \mu^P} \sum_{i=1}^{q} \sum_{j=1}^{q} |y_i - y_j|, \tag{3}$$

where μ^P is the mean income of the poor.

All three indices provide important information on poverty, but each of them alone has severe disadvantages. H is not sensitive to the intensity of the poverty per person, I is insensitive to the percentage of poor people and G is just an indicator for inequality among the poor.[5]

Based on a set of axioms, Sen (1976) developed the following index by combining the three indices presented above:[6]

$$S = H[I + (1 - I)G]. \tag{4}$$

This index, which we will refer to as the Sen-Index S, aggregates the information from the three single measures by multiplying the head-count ratio H with the sum of the income gap ratio I and the Gini-Index G weighted by $(1 - I)$.[7]

The rationale of the aggregation of I and a weighted G in this index can be understood as follows:[8] An increase in the inequality of the distribution of income among the poor with an unchanged average gap is in some way equivalent to an increase in the average gap without a change in the distribution. This can be expressed by an equivalent absolute shortfall which is computed as the normalized absolute deprivation represented by I plus the additional equivalent which captures the effect of inequality among the poor, $(1-I)G$. The product of this aggregate equivalent shortfall $[I + (1 - I)G]$ with the head-count measure H gives the Sen-Index as presented above.

3 Transfer of the Sen-Index into the Context of Shortfall-Risk Measurement

Some changes are necessary to make the Sen-Index useful for shortfall-risk measurement. The most important differences between income and yield distributions that have to be taken into account are (1) the need to reformulate the index for the application in a continuous framework and (2) the possibility of negative yields.

---

5 Cf. Sen (1976), p. 223.
6 Cf. Sen (1976) p. 223.
7 Alternatives and variations of this index by Sen himself can be found in Sen (1979).
8 Cf. Sen (1979), p. 299.

3.1 Adaptation to the use of continuous probability distributions

In this section, we adapt the Sen-Index to the framework of continuous probability distributions while we keep the restriction to positive yields. First, we transfer the single indices into the new framework, then we combine them into an aggregate measure analogously to the original Sen-Index.[9]

A shortfall-risk index maps a pair $(F(\cdot), z)$, where $F(\cdot)$ is an element of the set of probability distributions on $\mathbb{R}$ and the target yield $z \in \mathbb{R}$ into $\mathbb{R}$.[10]

For a random variable Y, $F_Y(y)$ is defined as the probability of the realization of Y not exceeding y: $F_Y(y) = P(Y \leq y)$. The value $F_Y(z)$ gives the probability of a yield realization that is not greater than z.[11]

The risk measure equivalent to the head-count ratio from poverty measurement for a given poverty line z is the value of the distribution function at the target yield z:

$$H^c(z) = F(z). \tag{5}$$

The function

$$F_m(z) \equiv \int_{-\infty}^{z} (z-y)^m dF(y) \tag{6}$$

defining risk in Fishburn (1977) is a formal representation of *Lower Partial Moments* of degree m.[12] By setting m to be zero, the *Lower Partial Moment Zero* (LPM_0) is a special case of this general function.[13]

It is easy to see that the LPM_0 for a given target z is equal to $F(z)$. The *type* of information is somewhat similar to the Value at Risk mentioned earlier. Our substitute for the head-count ratio is the probability of shortfall from a given target yield z, whereas the VaR is a yield realization that will be fallen short of with a given probability α.[14] Formally, assuming that F^{-1} is well defined and α is a fixed number with $0 < \alpha < 1$, the Value at Risk can be expressed as follows:[15]

$$VaR_\alpha(F_Y, z) \equiv max\{0, z - F^{-1}(\alpha)\}. \tag{7}$$

One characteristic inherent in both measures is the inability to cover the distribution of yields for all realizations worse than z or falling short of the VaR respectively. Hence, further information on the part of the distribution identified as comprising the shortfall events is needed.

---

[9] To emphasize the use of continuous distributions in this section, the indices are marked with the superscript c.

[10] Unlike in this general definition, in the remainder of this section the probability distributions and the poverty lines are defined on nonnegative real numbers $\mathbb{R}_+$. This restriction will be lifted in the next section. The indices are defined for $F(z) > 0$ on the part of the probability distribution function on $F(y)$ which is strictly increasing.

[11] In the sequel, the subscript Y is dropped for simplicity.

[12] We assume that F(y) is continuous and differentiable. Hence, the formula can be transformed into the form of a Riemann-Integral: $F_m(z) = \int_{-\infty}^{z}(z-y)^m f(y)dy$.

[13] Fishburn (1977), p. 116, imposed the restriction $m > 0$ in his paper.

[14] The level of confidence $(1-\alpha)$ is often chosen as 0.99.

[15] Cf. Jorion (1997), pp. 87.

The normalized average shortfall (if falling short) is measured by our substitute for the income gap ratio:

$$I^c(z) = 1 - \frac{\int_0^z yf(y)dy}{zF(z)}. \tag{8}$$

The numerator of the fraction above is an expression for the expected value of the realizations not exceeding the target z. The fraction as a whole gives the expected relative attainment of the target z with regard to the shortfall events. Subtracting this from 1 leads to the yield *gap* ratio. $I^c(z)$ would take on the value 1 if every outcome was zero. The absolute average gap can be expressed as $zI^c(z)$.

For the final component of the new measure, we turn to the transfer of the Gini-Index into the continuous framework. We choose the Lorenz-Curve as the basis for the computation of the Gini-Index because it also provides a graphical display of the dispersion of possible yields. In inequality measurement, it plots the proportion of total income in a community received by a bottom percentage of the population. From its definition, the Lorenz-Curve can take on values between zero and one and is increasing and convex. In the case of an equal distribution of incomes among the poor, the graph would be linear with a slope of one. The Gini coefficient is the ratio of the area between the Lorenz-Curve and the line of equal distribution (LED) and the area underneath the LED which is 0.5 by definition. In other words, the Gini coefficient is one minus twice the area underneath the Lorenz-Curve. To measure the inequality of a continuous probability distribution, the Lorenz-Curve is defined implicitly as follows:[16]

$$L^c(t) = \frac{1}{\mu}\int_0^{F^{-1}(t)} yf(y)dy, \tag{9}$$

with $t \in [0,1]$ denoting a bottom percentage of the yield realizations and μ denoting the mean of the distribution. Equivalently it can be written as:

$$L^c(t) = \frac{\int_0^{F^{-1}(t)} yf(y)dy}{\int_0^{F^{-1}(1)} yf(y)dy}. \tag{10}$$

Using this definition of the Lorenz-Curve, the Gini-Index for continuous probability distributions G^c can be written as:

$$G^c(F) \equiv 1 - 2\int_0^1 (t - L^c(t))dt = \frac{0.5 - \int_0^1 \frac{\int_{F^{-1}(0)}^{F^{-1}(t)} yf(y)dy}{\int_{F^{-1}(0)}^{F^{-1}(1)} yf(y)dy} dt}{0.5}. \tag{11}$$

Next, Lorenz-Curve and Gini-Index have to be modified in order to measure the inequality of the yield distribution for $y \leq z$ only. For the following presentation, the superscript D marks the indices defined on the downside part of the distribution which is comprising the shortfall events. Since all formulas following in this section are defined for continuous distributions, the superscript c is dropped now for simplicity.

For our purpose of measuring shortfall risk, the Gini-Index is defined for the shortfall part of the distribution. The index has to be normalized with respect to the distribution

[16] Cf. Atkinson (1970), p. 246. We assume that the inverse function $F^{-1}(y)$ exists.

of the shortfall events. We define the function $\tilde{F}(y) = \frac{F(y)}{F(z)}$ for $y \in [0, F(z)]$. By this transformation, the probability density function is also altered to $\tilde{f}(y) = \frac{f(y)}{F(z)}$ for $y \in [0, z]$. Hence, the Lorenz-Curve L^D is defined for $t \in [0, 1]$ as:

$$L^D(t) = \frac{1}{\mu^D} \int_0^{\tilde{F}^{-1}(t)} y\tilde{f}(y)dy, \tag{12}$$

where μ^D denotes the mean of the shortfall part of the distribution. For a given target z, the Gini-Index of the shortfall part of the probability distribution F can then be written as:

$$G^D(F, z) = \frac{0.5 - \int\limits_0^1 \frac{\int_{\tilde{F}^{-1}(0)}^{\tilde{F}^{-1}(t)} y\tilde{f}(y)dy}{\int_{\tilde{F}^{-1}(0)}^{z} y\tilde{f}(y)dy} dt}{0.5}. \tag{13}$$

The right-hand term of the numerator expresses the area underneath the Lorenz-Curve (displayed in Figure 1 below). The Gini-Index of the shortfall events is the ratio of the (shaded) area between the line of equal distribution (LED) and the Lorenz-Curve and the area under the LED (0.5).

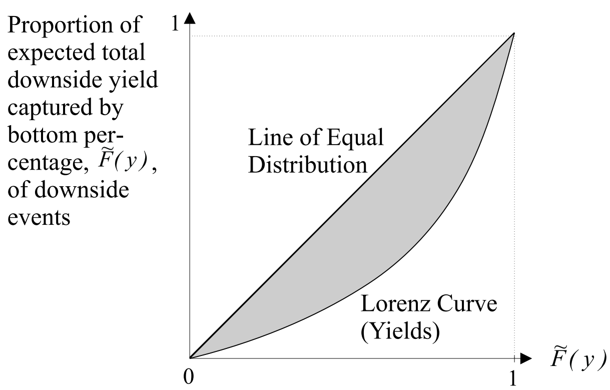

Figure 1: Lorenz-Curve and Gini-Index restricted to the shortfall events.

Cancelling the denominators of the normalized probability density functions and rewriting the limits of the integral that represents the area underneath the Lorenz-Curve yields:

$$G^D(F, z) = \frac{0.5 - \int\limits_0^1 \frac{\int_0^{\tilde{F}^{-1}(t)} yf(y)dy}{\int_0^z yf(y)dy} dt}{0.5}. \tag{14}$$

This shortfall Gini-Index G^D is used as the component for the measurement of inequality in the "new" Sen-Index, our measure for three dimensional shortfall risk:

$$S^D(z) = \underbrace{F(z)}_{H^c(z)} \left[\underbrace{1 - \frac{\int_0^z yf(y)dy}{zF(z)}}_{I^D(z)} + \left(1 - \left(\underbrace{1 - \frac{\int_0^z yf(y)dy}{zF(z)}}_{I^D(z)} \right) \right) G^D(z) \right]. \tag{15}$$

The index above is a representation of the Sen-Index in the context of shortfall-risk measurement. The underbraced parts are named after the single indices they represent. Rewriting yields:

$$S^D(z) = F(z)\left[\frac{\int_0^z (z-y)f(y)dy}{zF(z)} + \frac{\int_0^z yf(y)dy}{zF(z)}\left(1 - 2\int_0^1 t - \frac{\int_0^{\tilde{F}^{-1}(t)} yf(y)dy}{\int_0^z yf(y)dy}dt\right)\right]. \quad (16)$$

3.2 Adaptation in order to allow for negative yields

Since our purpose behind the transformation of the Sen-Index to the risk management framework was the measurement of the aforementioned three dimensions of shortfall risk, it has to be ensured that the individual indices integrated into the measure still work properly if yields can be negative.

Lifting the restriction of positive yields leaves our analogon for the head-count ratio, $F(y)$, unchanged. However, as shown in the next paragraphs the components for the indication of intensity and inequality of a possible shortfall, the yield-gap ratio and the Gini-Index, respectively, have to be modified.

3.2.1 Adaptation of the yield-gap ratio

If the yield-gap ratio is transferred to a context where negative realizations are possible by merely adapting its limits, for $F(z) > 0$ and $z \neq 0$ it can be expressed by:[17]

$$I^D(z) = 1 - \frac{\int_{-\infty}^{z} yf(y)dy}{zF(z)}. \quad (17)$$

Unfortunately, a closer look at this index unravels that when allowing for negative yields the reformulated index cannot be regarded as an appropriate measure for the intensity of the shortfall risk any more. The reason for this is that the structure of the measure causes serious problems, especially if the expected value of the shortfall yields, which we refer to as μ^D, is negative. Examining the possible values of the fraction in equation (17) under possible combinations of the signs of z and μ^D, the following cases are possible: [18]

1. z and μ^D are both positive. The measurement of the normalized average shortfall by (17) is uncritical in this case.
2. z is positive and μ^D is negative. In this case, the fraction takes on a negative sign, which would result in $I^D > 1$. This makes a reasonable economic interpretation impossible. Moreover, it makes the use of $(1 - I^D)$ as a weight for the Gini-Index in the aggregate questionable.

[17] It is assumed that the expected value of the shortfall distribution, as expressed in the numerator, does exist.

[18] Since z is the upper bound of the realizations in the shortfall part of the distribution, the combination of a negative z and a positive μ^D is not possible.

3. z and μ^D are both negative which implies a positive value of the fraction. Since the absolute value of the numerator is greater than that of the denominator, the fraction is greater than one and, therefore, I^D is negative. Again, a reasonable economic interpretation is not possible.

4. z is zero. This case is excluded from equation (17), but unlike in the context of poverty measurement, for the purpose of downside-risk measurement zero might be an appropriate value of z. Hence, the definition of the measure should be changed in a way that enables its use in this case.

The problems found above are due to (1) the normalization of the index with reference to a maximum shortfall z although shortfalls are not necessarily limited and (2) the possibly different signs of z and μ^D. To resolve these shortcomings, we perform a modification of the yield gap measure I^D, where y_{min} is the worst yield possible (or at least conceived to be the lowest one possible):

$$I^{D*}(z) = \frac{\int_{y_{min}}^{z} (z-y) f(y) dy}{(z - y_{min}) F(z)} \tag{18}$$

$$\Leftrightarrow I^{D*}(z) = \frac{\int_{y_{min}}^{z} (z-y) \tilde{f}(y) dy}{(z - y_{min})}. \tag{19}$$

In essence, the numerator in equation (18) resembles the LPM_1 (cf. (6)). If we impose the restriction of the maximal negative yield to be minus 100 percent, the normalized average shortfall can be expressed by

$$I^{D*}(z) = \frac{\int_{-1}^{z} (z-y) \tilde{f}(y) dy}{(z+1)}. \tag{20}$$

While a restriction like this induces some loss of generality, there is still a wide field of possible applications where such a restriction might be an appropriate projection of reality. Examples would be the cases where measurement is focused on portfolio yields only and short positions are not admissible or an application where single short positions in a portfolio are possible but only as hedge positions.

3.2.2 Adaptation of the Gini-Index

The limits of the integral in the Gini-Index are set by values of the inverse function of the transformed probability distribution, $\tilde{F}^{-1}(y)$. Therefore, the Gini-Index also captures negative realizations and could remain unchanged for at least formally comprising negative yields.

We now focus on the properties of a Gini-Index in the form represented in equation (14). For a graphical illustration we plot different types of Lorenz-Curves in Figure 2. At the point where 100 percent of the distribution in the downside part have been cumulated, and therefore 100 percent of the expected downside yield (EDY) have been reached, the point on the Lorenz-Curve is $(1, 1)$ by definition, independent of whether the EDY is negative or positive. Regarding the possible shape of the Lorenz-Curve, on which the Gini-Index is based, there is a difference to the shape shown in Figure 1 when allowing

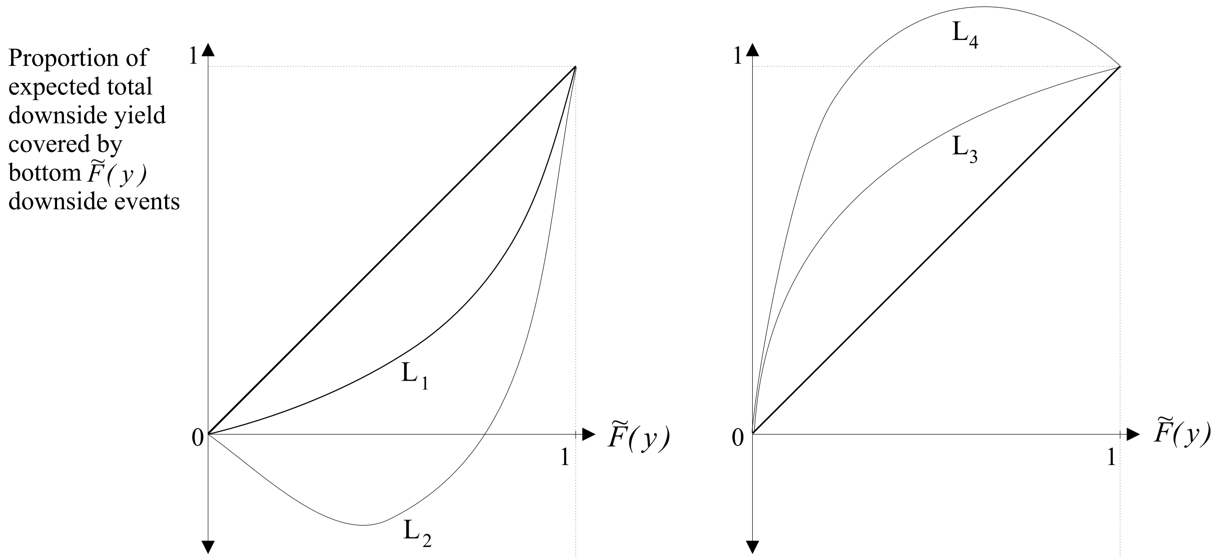

Figure 2: Lorenz-Curve types in the case of possibly negative yields

for negative yields. If the Lorenz-Curve for the shortfall events is modeled according to equation (12) for EDY $\neq 0$, one can distinguish four basic shapes which depend on the signs of the EDY and z:[19]

L_1 : The EDY is positive and no negative yields occur. This is the shape of the Lorenz-Curve which section 3.1 is restricted to.

L_2 : The EDY is positive, but negative yields may occur. For values of $F(y)/F(z)$ corresponding to the part of the distribution which is comprising only negative yields, the values of the Lorenz-Curve are also negative and the Lorenz-Curve is decreasing until y becomes nonnegative. The sign of the curve remains negative for a part of the distribution covering positive yields also. The values of the Lorenz-Curve are positive from the point on where the cumulated expected value of positive realizations compensates the cumulated expected value of the negative realizations.

L_3 : The EDY is negative and all shortfall yields are negative.

L_4 : The EDY is negative, but the shortfall part of the distribution also comprises positive yields. At the point where the slope of the Lorenz-Curve is zero, the proportion of expected shortfall yields covered is greater than 1. Aggregating over the remaining positive yields lowers the value of the curve to 1.

Notice that $z > 0$ in cases L_1, L_2, and L_4; as before (cf. Fn. 17) $z \leq 0$ implies EDY ≤ 0 (case L_3). The Lorenz-Curve runs on or above the LED in the cases L_3 and L_4 which would have to be accounted for in a new formula in order to accurately compute the area between Lorenz-Curve and LED. Greater problems may arise in the cases L_2 and L_4 since the value of the resulting Gini-Index may be very large in absolute value. In order to avoid these problems, the Gini-Index can be modified while keeping its intuitive appeal.

[19] The Gini-Index from equation (14) is not defined for EDY $= 0$. The new definition of the Gini-Index based on yield gaps (to be presented later in this section) resolves this problem.

Rearranging the usual formulation of the Sen-Index stated in detail in equations (15) and (16),

$$S^D = H^D[I^D + (1 - I^D)G^D], \tag{21}$$

to

$$S^D = H^D I^D \left[1 + \frac{(1 - I^D)G^D}{I^D}\right] \tag{22}$$

provides the basis for an alternative interpretation.[20] The Gini-Index as stated for poverty measurement in equation (3) computes one half of the relative mean difference. Clark et al. (1981) show that the following holds for the case of positive incomes:

$$\frac{(1 - I)}{I} = \frac{\mu_P}{\bar{g}}, \tag{23}$$

where $\bar{g}$ is the mean income gap and μ_p the mean income of the poor. This is not surprising since the mean income of the poor is the "complement" of the mean income gap. In addition to this, it is known that the difference in incomes between any two poor individuals equals the difference in their individual income gaps. By multiplying the definition of the Gini-Index by $\mu_P/\bar{g}$ and substituting the difference in income gaps for the difference in incomes in equation (3), it can be shown that the Gini-Index multiplied by $(1-I)/I$ is the Gini-Index of the distribution of income gaps which we denote by G*:[21]

$$\frac{(1 - I)G}{I} = G^*. \tag{24}$$

Therefore, the inequality of the distribution of shortfall events can be captured by a modified Gini-Index which is based on the distribution of yield gaps:

$$G^{D*}(F, z) = \frac{\int_0^1 \frac{\int_{\tilde{F}^{-1}(0)}^{\tilde{F}^{-1}(t)} (z-y)\tilde{f}(y)dy}{\int_{\tilde{F}^{-1}(0)}^{z} (z-y)\tilde{f}(y)dy} dt - 0.5}{0.5}. \tag{25}$$

Again, the index is normalized upon the distribution of shortfall events by use of $\tilde{F}(y)$ and $\tilde{f}(y)$. The shape of the new Lorenz-Curve is displayed in Figure 3. Unlike before, the Lorenz-Curve always runs above the line of equal distribution and has a decreasing slope. This representation of the Gini-Index can also be applied if EDY = 0, a case we excluded above. The computation of the Lorenz-Curve in this formulation is based on the expected yield gap which is different from zero if any downside events are possible.

Equation (23) does not hold if negative yields are admitted; the use of the same idea in our framework is nevertheless possible.[22] Inserting equation (24) into (22) gives an alternative formulation of the Sen-Index. The Gini-Index used in this formula is a measure of the relative shortfall in aggregate:

$$S^D = H^D I^D (1 + G^{D*}). \tag{26}$$

[20] Cf. Clark et al. (1981), p. 518-519.

[21] That is $\frac{(1-I)G}{I} = \frac{1}{2q^2\bar{g}} \sum_{i=1}^{q} \sum_{j=1}^{q} |g_i - g_j|$, cf. Clark et al. (1981), p. 519.

[22] For the case of negative yields with a lower bound, the complement to the yield gap has to be defined differently. The relation then holds in the form: $\frac{1-I}{I} = \frac{\mu_p - y_{min}}{\bar{g}}$. The effect of this, which seems to imply the use of a transformed distribution, has to be addressed in a more detailed axiomatic evaluation of the measure.

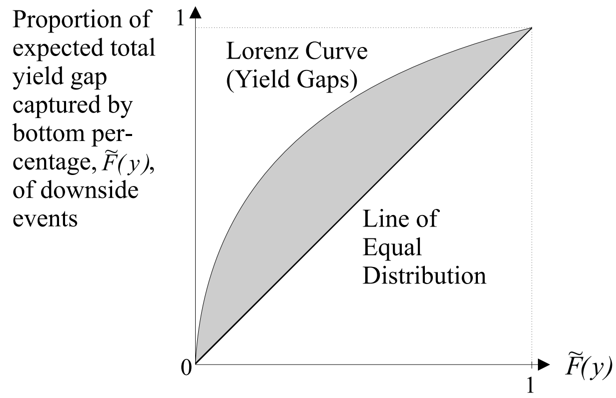

Figure 3: Lorenz-Curve for the Distribution of Yield Gaps

3.2.3 Formulation of the new Sen-Index

By employing this Gini-Index of yield gaps G^{D*} and the yield-gap ratio I^{D*} in the transferred and rewritten Sen-Index (22), it becomes possible to apply the new Sen-Index as a shortfall-risk measure for continuous probability distributions when yields can be negative:

$$S^{D*}(z) = F(z)\left(\frac{\int_{y_{min}}^{z}(z-y)\tilde{f}(y)dy}{z-y_{min}}\right)\left(2\int_{0}^{1}\frac{\int_{y_{min}}^{\tilde{F}^{-1}(t)}(z-y)\tilde{f}(y)dy}{\int_{y_{min}}^{z}(z-y)\tilde{f}(y)dy}dt\right). \tag{27}$$

The inclusion of the three different single indices into the aggregate measure formally accounts for the three dimensions of shortfall risk emphasized in this paper. For the context of poverty measurement, the performance of the Sen-Index, among other indices, in the light of an elaborated set of desired properties has been examined by Zheng (1997).

In the following section, we examine some characteristics of the new measure. We focus the evaluation of the measure presented above on one undesirable property. That shortcoming shall be overcome later in the present paper by means of another modification of the measure.

4 Selected Properties of the New Sen-Index from an Axiomatic Viewpoint

For the evaluation of possible benefits and shortcomings of the shortfall-risk index derived above, an axiomatic approach can be applied. For this purpose, an extensive axiomatic framework has been presented by Breitmeyer et al. (1999).[23] The set of axioms presented therein represents a catalog of possible properties one could demand to be fulfilled by a good shortfall-risk index. However, these properties cannot be regarded as necessary requirements for all kinds of applications nor for all types of investors. This is not surprising, since it may well be the case that regulatory authorities, for example, lay emphasis on properties which an investor deems secondary. Consequently, for any particular appli-

[23] For the context of poverty measurement, cf. Zheng (1997); Pattanaik and Sengupta (1995).

cation the set of important properties of the measure used - and by this the corresponding axioms - has to be determined separately. Nevertheless, there are axioms that are appropriate requirements for most applications. After presenting a set of basic axioms, in this section we mainly examine the transferred Sen-Index with respect to one particular axiom and highlight an undesirable feature inherent in this very measure which can be overcome by the modification developed in the next section. To keep the presentation short, our discussion is confined to an informal examination of the axioms discussed.

We start with three axioms that define basic requirements for a shortfall-risk measure. The basic axioms to be checked first are the following:

1. **Focus**: This axiom demands that the shortfall-risk index be sensitive to changes in the distribution below z only. Regarding the components of the modified Sen-Index, H^D states the probability of realizations $y \leq z$, the yield-gap ratio I^D and the Gini-Index G^D are both normalized on and restricted to realizations in the shortfall part of the distribution. Hence, S^D satisfies the Focus Axiom.

2. **Normalization**: This axiom requires the shortfall-risk index to be zero if no yield realizations below z are possible, i.e. if $F(z) = 0$. For the Sen-Index $H^D = F(z)$, thus S^D fulfills the Normalization Axiom.

3. **Non-Negativity**: This axiom demands that the shortfall-risk index be positive for all $F(z) > 0$. Given that $H^D = F(z)$ and that the target gap ratio I^D as well as the Gini-Index G^D can only take on values in [0;1], the index cannot become negative. Thus, S^D satisfies the Non-Negativity Axiom.

Having discussed these basic axioms, which are formulated to define minimum characteristics for a *shortfall-risk measure*, we now examine if the **Weak Transfer Sensitivity** Axiom is satisfied. This axiom is among the most interesting in the context of poverty-measurement and for several applications in shortfall-risk measurement it should be equally important.

The Weak Transfer Sensitivity axiom covers elements of both, the class of transfer axioms and the class of sensitivity axioms. In short, all others being equal, to satisfy the transfer axioms a shortfall-risk index has to increase when probability mass is transferred to a lower part of the yield distribution and to decrease when conversely probability mass is transferred to an upper part of the distribution. In refining the transfer axioms, the sensitivity axioms state to which extent a shortfall-risk index should be sensitive to equally sized distribution shifts which take place at different parts of the yield distribution.

The Weak Transfer Sensitivity Axiom demands that a shortfall-risk index be the more sensitive to equally sized shifts of probability mass the lower in the yield distribution they occur. I.e., the higher its shortfall is, the more weight is given to the respective event in the aggregate. For the discussion of the Weak Transfer Sensitivity Axiom with respect to S^{D*} we use the concept of Mean Preserving Spreads (MPSs) as stated by Rothschild and Stiglitz (1970). Analogously to MPSs we refer to Mean Preserving Contractions (MPCs), which are mean preserving shifts of probability mass from a distribution's "tails" to its center. For the context of our argumentation, we consider only MPSs and MPCs which are limited to the shortfall part of the distribution.

According to the Transfer Axioms, an MPS has to increase the shortfall-risk index whereas an MPC has to decrease the shortfall-risk index. To fulfill the Weak Transfer Sensitivity

Axiom, this change in S^{D*} must be the stronger the lower in the yield distribution the MPS or MPC respectively takes place.

Since both, an MPS and an MPC (both below z), leave the target gap ratio I^{D*} as well as the shortfall probability H^D unchanged, their impact on S^{D*} is dependent on the Gini-Index G^{D*} only. It is well known from the theory of inequality measurement that owing to the mechanism of the Lorenz-Curve, which the Gini-Index is based on, a rank order weighting system is underlying G^{D*}. It is a feature of a system of rank order weights that the weighting difference between certain events corresponds to the difference in the rank order of the events considered. In a discrete context with q events this translates to: The impact of the largest poverty gap on the Gini-Index is q times as strong as that of the smallest poverty gap. Correspondingly, the impact of the second largest poverty gap is $(q-1)$ times as strong as that of the smallest poverty gap. In a continuous context however, this rank order weighting system is implicitly established by means of $F(y)$, in that, loosely speaking, an "event"'s impact on the Gini-Index is determined by $1-F(y)$. Hence, the weighting difference of two events, y_1 and y_2, is given by $(1-F(y_1))-(1-F(y_2)) = F(y_2)-F(y_1)$. Against this background, and everything else being equal, an MPS's and an MPC's impact on the Gini-Index depend only on the weighting difference of the "events" affected by the respective MPS and MPC and not on where in the yield distribution they occur.

Summing up, S^{D*} does not satisfy the Weak Transfer Sensitivity Axiom. Neither H^D nor I^{D*} are affected by an MPS or an MPC respectively. However, both kinds of shift of probability mass would be detected by the Gini-Index. But all others being equal, G^{D*} is not sensitive to where in the downside tail of the distribution an MPS (or an MPC) is carried out.

5 Inclusion of Marginal Gap-Evaluation into the Sen-Index

The most important finding in the preceding section was that the new Sen-Index as its predecessor does not fulfill the Weak Transfer Sensitivity Axiom. To overcome this shortcoming, it is natural to modify components of the index. Our main purpose here is the construction of an index which shows increasing sensitivity to shifts of probability mass the lower in the distribution they occur. To achieve this, following Clark et al. (1981), we integrate the component which captures inequality, G^{D*}, into a framework similar to the marginal utility concept. However, unlike Clark et al. (1981), who make use of group social welfare functions for the measurement of relative income inequality,[24] we now employ gap-evaluation functions in order to measure relative shortfall inequality.

Applying this concept, we proceed in four steps. As a first step, the severity of shortfall in every single event is measured by so-called gap-evaluation functions. In a second step, the gap-evaluation (negative utility) of the shortfalls measured by the aforementioned gap-evaluation function is successively aggregated over all events. In a third step, a corresponding safety equivalent is calculated and divided by the expected shortfall of the existing yield distribution. Finally, the resulting ratio is then integrated into S^{D*} as a measure of inequality.

[24] Cf. Clark et al. (1981), pp. 518-523.

First, we assume a gap-evaluation function of the form

$$r_\phi(z-y) = -\left(\frac{1}{\phi}\right)(z-y)^\phi \tag{28}$$

with $\phi \in]1,\infty[$ as an inequality aversion parameter indicating that the negative marginal gap value is increasing in gaps.

Secondly, the gaps in the downside tail of the yield distribution function can be captured and evaluated by:

$$E[r_\phi(z-y)] = -\int_{y_{min}}^{z}\left(\frac{1}{\phi}\right)(z-y)^\phi f(y)dy. \tag{29}$$

Like in Clark et al. (1981), this aggregate evaluation function is symmetric, additive, and decreasing in gaps. The "disutility" is increasing in aggregated gaps, respectively. The inequality aversion parameter here serves as a measure of relative sensitivity of $E[r_\phi(z-y)]$ to gaps of different sizes. E.g., for $\phi = 1$ an investor's utility from an expected distribution of downside yields depends only on the expected shortfall, while for $\phi > 1$ his utility is the more determined by larger $(z-y)$ the higher ϕ is.[25] In the extreme, for $\phi \to \infty$, only the largest $(z-y)$ matters and portfolio choice according to $E[r_\phi(z-y)]$ turns into maximin with respect to yield.

For our purpose of measuring inequality in the distribution of shortfalls, we now calculate out an *equally distributed* equivalent shortfall $(z-y)^*$ that yields the same gap value (or negative utility) as the actually existing level and distribution of shortfall. For this, the condition

$$r_\phi[(z-y)^*] = E[r_\phi(z-y)] = -\int_{y_{min}}^{z}\left(\frac{1}{\phi}\right)(z-y)^\phi f(y)dy \tag{30}$$

must hold. This safety equivalent can be calculated, after some obvious steps, from

$$(z-y)^* = \left[\int_{y_{min}}^{z}(z-y)^\phi f(y)dy\right]^{\frac{1}{\phi}}. \tag{31}$$

Since the inequality in the distribution of shortfall is captured by $\frac{(z-y)^*}{E(z-y)} > 1$, the shortfall risk can then - analogously to formula (26) - be measured as

$$S^{D**}(z) = H^D I^{D*}\frac{(z-y)^*}{E(z-y)} \tag{32}$$

$$\Leftrightarrow S^{D**}(z) = H^D I^{D*}\frac{[\int_{y_{min}}^{z}(z-y)^\phi f(y)dy]^{\frac{1}{\phi}}}{\int_{y_{min}}^{z}(z-y)f(y)dy}. \tag{33}$$

The adequacy of $\frac{(z-y)^*}{E(z-y)}$ as a measure of inequality can be made clear for the case of two events by reference to Figure 4, where E_1, E_2 and E_1^*, E_2^* respectively represent gaps in two possible events.[26] Both (g_I, g_{II}) and (g_I^*, g_{II}^*) denote different sets of gap realizations, with the corresponding gap of each event to be read at the respective axis.

[25] Since our goal is to heal the violation of the Weak Transfer Sensitivity Axiom, we set $\phi \in]1,\infty[$ in order to express risk aversion. Nevertheless, ϕ can take on any other value to capture other attitudes towards risk.

[26] For similar inequality indices in welfare economics cf. Blackorby and Donaldson (1978).

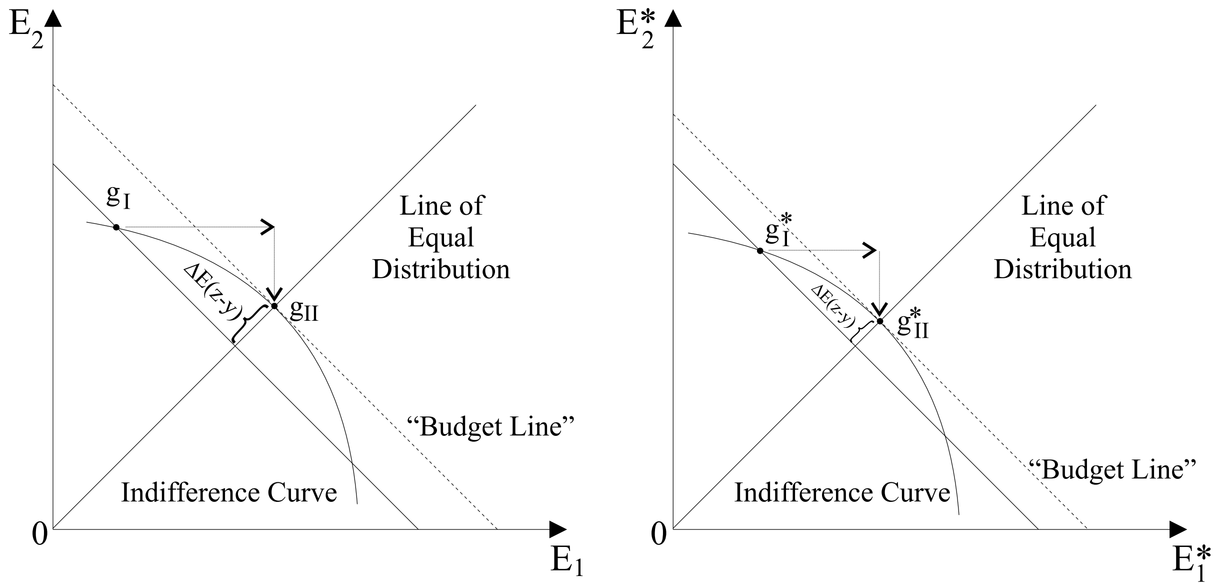

Figure 4: Utility Based Inequality Measurement

It is shown that owing to a change of existing gap realizations (g_I and g_I^* respectively) towards the line of equal distribution (towards g_{II} and g_{II}^* respectively) a certain level of utility, characterized by a certain indifference curve, can be reached with a larger expected shortfall (a lower expected yield). The increase in expected shortfall (the decrease in expected yield) $\Delta E(z-y)$, equalizing $r_\phi[(z-y)^*]$ and $r_\phi[E(z-y)]$, is the larger the more unequal the existing gap realizations (g_I and g_I^*) are distributed. Hence, $\frac{(z-y)^*}{E(z-y)} = \frac{E(z-y)+\Delta E(z-y)}{E(z-y)}$ can be applied as a measure of inequality in gap distributions.

By means of integrating the aforementioned gap-evaluation function in our measure of inequality, which substitutes the Gini-Index, our shortfall-risk index now satisfies the Weak Transfer Sensitivity Axiom. The reason behind this is the fact that the inequality aversion parameter is greater than 1, which gives increasing weight to yield gaps. This increase is more than proportional. Thus, S^D becomes the more sensitive to equally sized shifts of probability mass the lower in the yield distribution they occur. The same effect could be reached by substituting the Gini-Index by a somehow normalized LPM_2. However, this would include the assumption of a specific degree of risk aversion while the variation of $\phi \in]1, \infty[$ allows the consideration of different degrees of risk aversion.

This is an example for a possible modification of the Sen-Index in order to increase its sensitivity to very bad events. However, any modification of the index also should pass an examination with respect to a greater number of appropriate axioms prior to implementation.

6 Conclusion

For the measurement of the three dimensions incidence, intensity, and inequality of possible shortfalls, the Sen-Index has been transferred from the context of poverty measurement into the framework of shortfall risk. To build a shortfall-risk index which maps a probability distribution and a target yield into an aggregate index analogously to the originial

Sen-Index, the restriction that a lower bound of the portfolio yields has to be known was imposed.

The new measure that has been developed is capable of accounting for the three dimensions of shortfall risk in an aggregate. By this it makes possible a ranking of different distributions with respect to all three dimensions jointly. If the information from the three individual indices included in the Sen-Index is also displayed separately, details on the characteristics of the distributions involved can give hints as to why a specific ranking was assigned. This could be very helpful, especially in the case where none of the examined portfolios matches the investor's demands or fulfills given risk restrictions.

Notice that there exist other indices which can be represented as aggregates of the head-count ratio, the income gap ratio, and an inequality index (cf. Zheng (1997), p. 24-35). Kakwani (1999) advocates such measures and also suggests axioms (challenged by Ravallion (1999)) to determine how such an aggregation should be done.

While the explicit aggregation of different indices as presented herein is somewhat new in shortfall-risk measurement, the representations of the single indices chosen for the construction of a Sen-Index in the new framework are well known and partly already used in risk management. The Lower Partial Moments Zero and One are explicitly incorporated in the measure. It has been shown that, captured by the former, information similar to the Value at Risk is contained in the measure.

Furthermore, an additional modification was presented in order to improve the new Sen-Index with respect to a specific shortcoming discussed in the brief axiomatic evaluation.

Admittedly, the assumptions underlying the way information on the distributions is aggregated in the transformed Sen-type measures might be discussed controversially. Depending on the field of application, a set of particularly important axioms has to be identified. Only after this has been done, an appropriate measure can be selected.

While the type of measure presented cannot claim the advantage of being a concept satisfying an exceptionally great number of possible axioms, it has special appeal because it not only incorporates the dimensions of risk highlighted but also makes them clearly visible. Therefore, the transferred Sen-Index may contribute to narrowing the gap in the measurement of shortfall risk that was outlined at the beginning of the paper.

References

Arrow, Kenneth J. (1963), Social Choice and Individual Values, 2nd ed., Wiley, New York, 1st ed. 1951.

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber and David Heath (1997), Thinking Coherently, *Risk*, vol. 10, pp. 68–71.

Atkinson, Anthony B. (1970), On the Measurement of Inequality, *Journal of Economic Theory*, vol. 2, pp. 244–263.

Blackorby, Charles and David Donaldson (1978), Measures of Relative Equality and Their Meaning in Terms of Social Welfare, *Journal of Economic Theory*, vol. 18, pp. 59–80.

BREITMEYER, Carsten, Hendrik HAKENES, Andreas PFINGSTEN and Christoph RECHTIEN (1999), Learning From Poverty Measurement: An Axiomatic Approach to Measure Downside Risk, Institut für Kreditwesen, Westfälische Wilhelms-Universität, Münster, Diskussionsbeitrag 99-03.

CLARK, Stephen, Richard HEMMING and David ULPH (1981), On Indices for the Measurement of Poverty, *The Economic Journal*, vol. 91, pp. 515–526.

DASGUPTA, Partha, Amartya SEN and David STARRETT (1973), Notes on the Measurement of Inequality, *Journal of Economic Theory*, vol. 6, pp. 180–187.

EGGERS, Frank, Sven RIESO and Andreas PFINGSTEN (1999), Aggregation von drei Dimensionen des Downside-Risikos durch einen modifizierten Sen-Index, in: Karl INDERFURTH, Gerhard SCHWÖDIAUER, Wolfgang DOMSCHKE, Friedrich JUHNKE, Peter KLEINSCHMIDT and Gerhard WÄSCHER (eds.), *Opereations Research Proceedings 1999 - Selected Papers of the Symposium on Operations Research (SOR '99), Magdeburg, September 1-3,1999*, Springer-Verlag, Berlin et al., pp. 311-316.

EICHHORN, Wolfgang and Arthur VOGT (1990), Gemeinsames bei der Messung von Ungleichheit, Streuung, Risiko und Information, *Mitteilungen der Schweizerischen Vereinigung der Versicherungsmathematiker*, no. 1.

FISHBURN, Peter C. (1977), Mean-Risk Analysis with Risk Associated with Below-Target Returns, *The American Economic Review*, vol. 67, no. 2, pp. 116–126.

GUTHOFF, Anja, Andreas PFINGSTEN and Juliane WOLF (1997), Effects on Risk Taking Resulting from Limiting the Value at Risk or the Lower Partial Moment One, in: *7th International AFIR Colloquium Proceedings*, Southwood Press, Sydney.

JENKINS, Stephen P. and Peter J. LAMBERT (1997), Three "I's of Poverty" Curves: TIPs for Poverty Analysis, *Oxford Economic Papers*, vol. 49, no. 3, pp. 317–327.

JOHANNING, Lutz (1998), Value-at-Risk zur Marktrisikosteuerung und Eigenkapitalallokation, Uhlenbruch, München.

JORION, Philippe (1997), Value at Risk, Irwin, Chicago.

KAKWANI, Nanak (1999), Inequality, Welfare and Poverty: Three Interrelated Phenomena, in: Jaques SILBER (ed.), *Handbook on Income Inequality Measurement*, Kluwer Academic Publishers, Boston et al., pp. 599–628.

NERMUTH, Manfred (1993), Different Economic Theories with the Same Formal Structure: Risk, Income Inequality, Information Structures, etc., in: W. Erwin DIEWERT, Klaus SPREMANN and Frank STEHLING (eds.), *Mathematical Modelling in Economics*, Springer, Berlin, pp. 271–277.

PATTANAIK, Prasanta K. and Manimay SENGUPTA (1995), An Alternative Axiomatization of Sen's Poverty Measure, *Review of Income and Wealth*, vol. 41, no. 1, pp. 73–80.

RAVALLION, Martin (1999), Comment on: Inequality, Welfare and Poverty: Three Interrelated Phenomena by Nanak Kakwani, in: Jaques SILBER (ed.), *Handbook on Income Inequality Measurement*, Kluwer Academic Publishers, Boston et al., pp. 629–634.

Rothschild, Michael and Joseph E. Stiglitz (1970), Increasing Risk I: A Definition, *Journal of Economic Theory*, vol. 2, pp. 225–243.

Rothschild, Michael and Joseph E. Stiglitz (1973), Some Further Results on the Measurement of Inequality, *Journal of Economic Theory*, vol. 6, pp. 188–204.

Sen, Amartya (1976), Poverty: An Ordinal Approach to Measurement, *Econometrica*, vol. 44, pp. 219–231.

Sen, Amartya (1979), Issues in the Measurement of Poverty, *Scandinavian Journal of Economics*, pp. 285–307.

Zheng, Buhong (1997), Aggregate Poverty Measures, *Journal of Economic Surveys*, vol. 11, pp. 123–162.