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The Economic Life of Industrial Equipment Reconsidered

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Abstract

The paper deals with two particular investment decisions: optimizing the investment date and optimizing a single identical replacement. The objective is to maximize the net present value. For the common and convenient assumption of a flat term structure of interest rates, two results are known. Firstly, it is never optimal to delay an investment with a positive net present value. Secondly, the optimal economic life of the first machine is less than or equal to the optimal economic life of the second machine ("general law of replacement"). We demonstrate that both results seize to hold when non-flat term structures are allowed. Examples are provided for inverse as well as normal term structures, and it is shown why the latter are necessarily more complicated. In a broader perspective, the paper proves that the assumption of a flat term structure is not innocuous and the seemingly general well-known results only hold for a special case.

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1 Motivation

Industrial Equipment of any kind is worn out by its use and eventually has to be replaced. Typically, a machine will not be replaced by an identical one. The reasons are manifold. Technical progress may make better machinery available, and often machines of previous vintages may not be produced and offered any more. Also, size and composition of a firm's production may have changed over time implying that the initial machine would not be optimal anymore. For an analytical examination, however, the replacement by an identical machine constitutes an important benchmark.

Firms are usually founded for an infinite life span. Consequently, a machine will be replaced more than once. To achieve accordance with the infinite planning horizon of a firm, one would have to analyze an infinite number of replacements. Since uncertainty usually increases significantly with time, a finite number of replacements is usually assumed with a finite time horizon given. Again as a benchmark case, a single replacement is considered.

Combining both aspects, we are discussing the single replacement by an identical machine. This issue is well analyzed and the reader may wonder what we have to add. A "general law of replacement" appears in the literature (cf. Preinreich (1953)) which dates back at least to Preinreich (1940) who states that "each machine will have a longer life than its predecessor and a shorter life than its successor in the chronological chain" (p. 17). We will show that this statement crucially depends on the term structure of interest rates which is used when making the initial investment decision. Moreover it will turn out that delaying an investment can even be beneficial, if a non-flat term structure is being used, without considering the embodied real options.

In Germany the "general law of replacement" is embraced by the terms "Ketteneffekt" or "Gesetz der Ersatzinvestition" in the standard literature to investment theory (cf. e.g. Adam (2000), Kruschwitz (2000) and Schneider (1992)). Apart from a controversy between Buchner (1980 and 1982) and Zechner (1981 and 1982) concerning the statements of Lutz (1951, p. 32-34 and 108-109) and Schneider (1975, p. 286-289) that the law is only of small importance for growing companies (and so for investment theory in general), its validity has not been a major point of discussion. Recently, Johannwille (2000) corrected some shortcomings of Rolfes (1998) who explicitly examined the influence of assumptions about the term structure of interest rates on the validity of the law.

Our note is organized as follows: In Section 2 we will model the investment decision to be discussed. In Section 3 we analyze under which circumstances it is beneficial to delay a single investment for one period. Section 4 then is devoted to the optimal decision in the case of a single, identical replacement with a fixed starting date ($t = 0$), whereas in Section 5 the starting date is optimized, too. Section 6 summarizes and indicates further applications of our findings.

2 Investment decisions

Any replacement decision is an investment decision. Therefore it is sensible to start with a framework in which such investment decisions are being made. Abstracting from many important real world problems, we will assume certainty with respect to cash flows. In other words, an investment project is completely represented by its net cash flows cf_0, cf_1, ..., cf_T, where T is the length of the project, i.e. the time of usage in case of a machine. Strictly speaking, the project has a sequence of cash flows which are zero after some date E. When a project (e.g. the usage of a machine) is terminated in some $T \leq E$, this means that all cash flows after T are equal to zero. It is possible that termination in T involves extra revenues or costs (e.g. sale of the machine or its scrapping) which may vary with T. We will abstract from this additional complication for the main part of our paper, but will return to it at the end. For simplicity, we will also assume that the sequence of cash flows is the same no matter at which time t the project is started. This assumption seems to be well-suited in cases of mass production where the complete output is being sold at time-invariant market prices and the variation in cash flows basically comes about via different user costs of the machine, including repair and the like. Needless to say that the assumption is inappropriate in many other cases.

Assuming perfect capital markets, it is standard practice (cf. e.g. Copeland and Weston (1988), p. 26 or Jaffe, Ross, and Westerfield (1999), p. 57) to use the NPV criterion when making investment decisions. According to this rule, a project is worth to be pursued if and only if its net present value is positive (or at least not negative). Of two mutually exclusive projects, the one with the higher NPV should be chosen.

The net present value is calculated by adding the cash flows discounted to the time when the decision is being made, say at time 0. The discounting is done under the (sometimes tacit) assumption of a flat term structure, i.e. a constant interest rate r. A brief look at real world capital markets reveals that a flat term structure is the exception rather than the general rule. Therefore it seems reasonable to allow for a non-flat term structure when discounting, too. This is particularly true when the investor wants to hedge against interest rate risk and therefore employs cash flow hedging, using the market interest rates. Calculating net present values with a non-flat term structure is well accepted when dealing with financial investments (cf. e.g. Copeland and Weston (1988), p. 66 or Brealey and Myers (2001), p. 36). Sometimes it is argued, however, that the relevant interest rates cannot be derived from the capital market directly, but mirror some sort of capital cost which is more or less constant over time. Obviously, we do not buy this argument.

We denote the discount factors by q_0, q_1, q_2, ..., where we assume $q_0 = 1$ and $q_\tau > q_{\tau+1} > 0$ for all $\tau \geq 0$, i.e. positive interest rates for all periods. In the special case of a flat term structure with interest rate r we have $q_\tau = (1 + r)^{-\tau}$. The net present value of an investment project started at time t with a length of T then is:

$$NPV_{t,T} = \sum_{\tau=t}^{t+T} cf_{\tau-t} * q_\tau. \quad (1)$$

3 Optimal investment date

This section is devoted to an analysis of the optimal timing decision for an investment project. More specifically, we ask ourselves whether it is always optimal to invest immediately or whether it can be better to wait for another period. Formally, we compare the net present value of the project started in $t + 1$ to the net present value of the project started in t:

$$\Delta NPV_{t+1,t} = \sum_{\tau=t+1}^{t+1+T} cf_{\tau-t-1} * q_\tau - \sum_{\tau=t}^{t+T} cf_{\tau-t} * q_\tau = \sum_{\tau=t}^{t+T} cf_{\tau-t} * (q_{\tau+1} - q_\tau), \quad (2)$$

with the term in brackets always being negative.

If $\Delta NPV_{t+1,t}$ is negative then it is better to invest in t than to wait until $t + 1$. In case of a flat term structure with an interest rate $r > 0$, waiting is never strictly preferred. Expanding the term in brackets for this case yields

$$q_{\tau+1} - q_\tau = \frac{1}{(1+r)^{\tau+1}} - \frac{1}{(1+r)^\tau} = \frac{1}{(1+r)^\tau} * \left(\frac{1}{1+r} - 1\right) = -\frac{1}{(1+r)^\tau} * \frac{r}{1+r}, \quad (3)$$

and by inserting into equation (**??**):

$$\Delta NPV_{t+1,t} = -\frac{r}{1+r} * \sum_{\tau=t}^{t+T} cf_{\tau-t} * \frac{1}{(1+r)^\tau} = -\frac{r}{1+r} * NPV_{t,T}. \quad (4)$$

It is obvious from equation (**??**) that the net present value of the delayed project is $1/(1 + r)$ times the net present value of the original project. Equation (**??**) then reveals that the difference in NPV is proportional to the NPV of the project. This implies:

Proposition 1[1]

For a flat term structure with interest rate $r > 0$, it is never optimal to delay a project with positive net present value.

Equation (??) tells us more about the possibility of a beneficial delay of projects. Waiting increases the net present value if and only if the absolute values of the (negative) terms in brackets are relatively large for negative cash flows and relatively small for positive cash flows.

A simple investment project is defined as one with $T = 1$ and $cf_0 < 0 < cf_1$. In case of an inverse term structure, i.e. interest rates decreasing with time to maturity, it can be shown that even simple investment projects give rise to situations where a delay for one period increases the net present value:

Example 1

Our first example is a simple investment with the cash flows $cf_0 = -100$ and $cf_1 = 113$. Given a flat term structure with an interest rate of 10%, $NPV_{0,1}$ is positive and $\Delta NPV_{1,0} = -100 * (0,90909091 - 1) + 113 * (0,82644628 - 0,90909091) \approx -0,2479 < 0$.

This shows that it is advantageous to start the investment in $t = 0$ in this case. The economic intuition behind this result is that, in absolute terms, the reduction in the present value of the return cf_1 is greater than the reduction in the (negative) present value of the initial investment outlay cf_0.

A different result can be observed if an inverse term structure is assumed, e.g. interest rates[2] $r_1 = 11\%$ for one year and $r_2 = 10\%$ for two years. The $\Delta NPV_{1,0}$ of the project is in this case $-100 * (0,90090090 - 1) + 113 * (0,82644628 - 0,90090090) \approx 1,4965 > 0$.

This example proves:

Proposition 2[3]

For an inverse term structure, it is possible that the positive net present value of a project is increased by waiting for one (or more) period(s).

The most difficult situation is the one which is probably the most realistic one, namely a so-called normal term structure. In this case, interest rates increase with time to maturity. It turns out that for simple projects as in the previous example a delay can never be optimal. The intuitive reason for a delay of one period is as follows: Shifting the project means to make the (positive) paybacks subject to the relatively small discount

[1] This result is probably informally known in the profession but does not seem to be well-documented as such in the literature.

[2] To keep the calculations simple we assume zero coupon spot rates in all examples.

[3] This observation is basically present in Johannwille (2000, p. 105-106), too, but for a slightly more complex project.

factor for the second period. In comparison, the initial investment expenditure is only mildly discounted. Therefore, a simple project with positive NPV will lose in value if it is delayed for one period. The general result and proof is as follows:

Proposition 3
Suppose a simple project has a positive NPV for $t = 0$ and the one-period interest rate $r_1 > 0$. Then it is necessary, but not sufficient, that the term structure is inverse for a start in $t \geq 1$ to generate a higher NPV.

Proof:
By assumption, $NPV_{0,1} > 0$. A necessary condition for a later start of the project to increase the NPV is $NPV_{t,1} > 0$ for some $t \geq 1$. This implies

$$\begin{aligned} NPV_{t,1} &= cf_0 * \frac{1}{(1+r_t)^t} + cf_1 * \frac{1}{(1+r_{t+1})^{t+1}} \\ &= \frac{1}{(1+r_t)^t} * \left(cf_0 + cf_1 * \frac{(1+r_t)^t}{(1+r_{t+1})^{t+1}} \right) \\ &< cf_0 + cf_1 * \frac{1}{1+r_{t+1}} \\ &< cf_0 + cf_1 * \frac{1}{1+r_1} = NPV_{0,1}, \end{aligned}$$

where the inequalities hold because of $r_{t+1} > r_t > 0$ and $r_{t+1} > r_1$ for $t \geq 1$. Therefore, in the case of a normal term structure no delay can be value-increasing for a simple project with positive $NPV_{0,1}$. ■

It turns out, however, that more complicated investment projects, in particular those where the signs in the series of cash flows change more than once, may yield higher net present values if delayed for one period even under a normal term structure. The following example is one of the easiest conceivable cases.

Example 2
An investment yields the cash flows $cf_0 = -100$, $cf_1 = 226$, and $cf_2 = -122$.
Assuming a flat term structure with an interest rate of 10%, the change in NPV of this project is $\Delta NPV_{1,0} = -100 * (0,90909091 - 1) + 226 * (0,82644628 - 0,90909091) - 122 * (0,75131480 - 0,82644628) \approx -0,4207 < 0$.
Waiting becomes advantageous if a normal term structure with $r_1 = 9\%$, $r_2 = 10\%$,

and $r_3 = 12\%$ is assumed. $\Delta NPV_{1,0} = -100 * (0,91743119 - 1) + 226 * (0,82644628 - 0,91743119) - 122 * (0,71178025 - 0,82644628) \approx 1,6836 > 0$.[4]

In this example, again, a shift is not beneficial under a flat term structure, but it is beneficial under some normal structure. Therefore we have shown:

Proposition 4

For a normal term structure, it is possible that the positive net present value of a non-simple project is increased by waiting for one (or more) period(s).

In particular the second example shows how special the "general" result, sub-optimality of delaying positive NPV projects, derived for the flat term structure actually is.

4 Optimal single identical replacement with a fixed starting date

The observations of the previous section are important in their own right. They also are the major ingredient to our findings in this section. Here, we challenge the generality of what seems to be adopted as the "general law of replacement". The famous statement of Preinreich (1940, p. 17) implies

Proposition 5

For a flat term structure and the case of a single identical replacement, the economic life of the second machine is greater than or equal to that of the first machine.

Firstly, let us intuitively explain why this holds for a flat term structure by deriving a contradiction. Suppose, the first machine was used $T + 1$ periods and the second machine only T periods. What would happen if we reduced the life of the first machine by one period? Given a flat term structure, the net present value contributed by the second machine would increase.[5] If the cash flow of the first machine in period $T + 1$ was negative, then the reduction of its time of use would further increase the net present value. Hence the initial solution was not optimal. Yet if the cash flow of the first machine in period $T + 1$ was positive, then it would not make sense to use the second machine only for T periods. Therefore in this case the initial solution could not have been optimal either. Summing up, under a flat term structure it cannot be optimal to use the first machine one period longer than an identical second machine.

[4] The reader should note that the inverse term structure $r_1 = 12\%$, $r_2 = 10\%$, and $r_3 = 9\%$ also makes a delay of this investment advantageous because $\Delta NPV_{1,0} = -100 * (0,89285714 - 1) + 226 * (0,82644628 - 0,89285714) - 122 * (0,77218348 - 0,82644628) \approx 2,3255 > 0$.

[5] We already know that this does not necessarily hold for non-flat term structures; cf. Section 3.

Secondly, we want to use the argument of the preceding paragraph to show what is different when the term structure is not flat. Suppose that T is the optimal time to use the second machine. In passing notice that this need not be the same irrespective of when the second machine is first employed.[6] Neglecting this possible complication for a moment, let us suppose that the first machine is also used for T periods. Suppose as well that the cash flow of the machine is negative in all periods past T. Then using the first machine for $T + 1$ periods would not make sense at all for a flat term structure, because both effects, negative cash flow in period $T + 1$ and delay of the use of the second machine, would work in the same direction, namely reducing total net present value. As we know from Section 3, however, for a non-flat term structure the delay of the use of the second machine can increase its contribution to total net present value. The additional present value generated by the shift to the next period can indeed overcompensate the negative cash flow of the first machine in period $T + 1$. Therefore we have informally shown the following result:

Proposition 6
If the term structure is non-flat, the optimal economic life of a machine maybe longer than that of its identical successor, if the initial investment has to start in $t = 0$.

Formally, this result is proved by the following example:[7]

Example 3
We examine an investment which is very similar to our first example and yields the following cash flows: $cf_0 = -100$, $cf_1 = 113$, and $cf_2 = -1$. Assuming a single replacement, on principal there are four possible combinations of the time of use (either one or two years for each machine). The example, however, has a special feature which facilitates the analysis and is permissible when looking for a (sufficient) counterexample: the optimal time of use of the second machine will obviously be equal to one, irrespective of its start, as long as the relevant one-period interest rate does not exceed 13% (in which case the second machine would not be of use at all). Therefore, we only need to compare the times of use for the first machine which can take values one or two, the latter being a contradiction to the "general law". With a sequence of projects, it is necessary to define how the sequencing of cash flows occurs. In particular we need to determine whether the initial outlay of the second project occurs in the same period as the final cash flow of the first period or one period later. We assume the first case, i.e. for a replacement after

[6] Rolfes (1998, p. 235-271) claims to provide a counterexample to the "general law" but neglects this important point. He proceeds retrograde, i.e. determines the optimal usage time of the second machine first, assuming the yield curve of $t = 0$. But in general under a non-flat term structure this yield curve does no longer prevail when the use of the second machine actually starts. Therefore the contribution of the second machine to the total NPV is not just its NPV calculated with the yield curve of $t = 0$ and discounted to $t = 0$ from its actual starting date. Consequently the given example is not really a valid counterexample.

[7] We assume, for the moment, that the initial investment has to start in $t = 0$. In Section 5 we will relax this assumption.

one year the total cash flow is $(-100; 113-100; 113)$ and not $(-100; 113; -100; 113)$. The following results do not depend on this assumption very much. Observations similar to ours can be constructed for the second case, too.

Table 1 shows the total NPV if a flat term structure with an interest rate $r = 10\%$ is presumed:

Table 1: Example 3 with a flat term structure

Time of use of machine 1	1	2
NPV	**5,2066**	4,1548

It is obvious that it is optimal to use both machines for only one year.

As can be seen from Table 2, the results change to an optimal time of use of two years for the first machine, violating the "general law", if we assume an inverse term structure with interest rates $r_1 = 11\%$, $r_2 = 10\%$, $r_3 = 9\%$, and $r_4 = 8\%$:

Table 2: Example 3 with an inverse term structure

Time of use of machine 1	1	2
NPV	5,1001	**5,5875**

The reader should note the result for the second way of sequencing, i.e. the cash flow from the second machine starting one period after the cash flow from the first machine has ended. There, the corresponding $NPVs$ are 6,4139 and 6,8154, respectively, implying the same optimum and proving Proposition 6 for an inverse term structure, too. Using a different example, we can also derive contradictions to the "general law" for a normal term structure when assuming that the initial investment has to start in $t = 0$:

Example 4
A machine can be used for up to three years with cash flows $cf_0 = -100$, $cf_1 = 110$, $cf_2 = -300$, and $cf_3 = 320$. In this example there are nine different possible combinations of the time of use for both machines. Table 3 shows the total $NPVs$ for each combination presuming a flat term structure with an interest rate $r = 5\%$.

Table 3: Example 4 with a flat term structure

Tou_1 / Tou_2	1	2	3
1	9,2971	-249,8542	13,4106
2	-263,0278	-509,8385	-259,1101
3	13,1946	-221,8632	**16,9257**

with Tou_z = Time of use of machine z.

In this case the total NPV is maximized by using both machines for three years each. A different optimum results from Table 4 where the total $NPVs$ are calculated for the normal term structure with interest rates $r_1 = 2\%$, $r_2 = 2,5\%$, $r_3 = 3\%$, $r_4 = 3,5\%$, $r_5 = 4\%$, and $r_6 = 4,5\%$.

Table 4: Example 4 with a normal term structure

Tou_1 / Tou_2	1	2	3
1	14,5035	-260,03900	18,8225
2	-272,2170	-533,6497	-270,6330
3	**19,4886**	-227,0895	18,6371

Assuming the given normal term structure, a time of use of three years for the first and only one year for the second machine becomes optimal.

The example underlying Table 4 proves Proposition 6 for a normal term structure. It is closely related to an example by Johannwille (2000, p. 109). Basically, it consists of a sequence of two simple projects. The forward interest rates implicit in the given term structure are chosen such that running the second part of the project rather than cancelling it is value increasing initially but not at the end of the second project. This can be seen from comparing the internal rates of return of the two parts, $IRR_1 = 10\%$ (for cf_0 and cf_1) and $IRR_2 = 6,66667\%$ (for cf_2 and cf_3), respectively, with the one-period implicit forward rates ($r_{c,d}$ = implicit forward rate from period c to period d) $r_{1,2} = 3,00245\%$, $r_{2,3} = 4,00733\%$, $r_{3,4} = 5,01461\%$, $r_{4,5} = 6,02427\%$, and $r_{5,6} = 7,03629\%$, where in particular $r_{5,6} > IRR_2$ reduces the optimal time of use of machine 2 from 3 to 1.[8]

5 Simultaneous optimization of investment date and single identical replacement

Until now we have assumed that the initial investment must start in $t = 0$. Given a flat term structure, this assumption is innocuous because the delay of a project with positive NPV cannot be optimal anyway (Proposition 1). For an inverse (normal) term structure, however, we know from Proposition 2 (4) that a delay may increase the (positive) NPV. Therefore forcing $t = 0$ as the start of the first project could induce a sub-optimal decision exogenously. Therefore we look at the joint optimization of date and duration of investments next.

[8] In passing, note that the example clearly proves the retrograde method of Rolfes mentioned in Footnote 6 is wrong: the optimal time of use of the second machine is 3 if started in $t = 1$ (14,5035 < 18,8225; cf. Table 4), but 1 if started in $t = 3$ (19,4886 > 18,6371).

We continue by using the data of Example 3. It turns out that, given the inverse term structure, it would indeed be beneficial if the project could start after $t = 0$. Let further interest rates be given by:

Table 5: Extended inverse term structure

r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}
7,00%	5,90%	5,50%	4,90%	4,80%	4,70%	4,69%	4,68%	4,67%	4,66%

and r_{15} to r_{20} follow the rule $r_t = r_{t-1}$ - 0,0001.

Let $NPV_t^{i,j}$ denote the *NPV* if the first machine is used i periods, the second machine j periods, and the sequence starts in t. Then we obtain the *NPVs* reported in Table 6.

Table 6: Example 3 with an extended inverse term structure

t	0	1	2	3	4	5	6	7	8	9
$NPV_t^{1,1}$	5,10	7,91	10,45	12,90	**15,88**	15,60	15,11	14,22	11,71	10,89
$NPV_t^{2,1}$	5,59	8,37	10,94	13,94	13,14	**16,45**	12,00	13,48	10,35	10,07

t	10	11
$NPV_t^{1,1}$	9,95	9,54
$NPV_t^{2,1}$	9,17	8,79

We see a couple of things:

Firstly, both sequences can generate higher *NPVs* if they need not start in $t = 0$. Note that the *NPVs* of the first sequence are increasing until $t = 4$ and decreasing afterwards, whereas for the *NPVs* of the second sequence such a monotonicity does not hold (since the optimum occurs at $t = 5$ but $NPV_7^{2,1} > NPV_6^{2,1}$).[9]

Secondly, it depends on the starting date which sequence generates the higher *NPV*. There is no monotonic behaviour in the sense that one sequence is preferred for $t \leq \bar{t}$ and the other for $t > \bar{t}$ (check t = 3, 4, 5, 6).

Thirdly, for high values of t ($t \geq 8$, say) both *NPVs* are decreasing and the first sequence dominates. There is good reason for that. Intuitively, the implicit forward rates approach values in the order of magnitude of the initial long-term interest rates.[10] Obviously, the

[9] Note that changing r_6 from 5,90% to 6,00% would make $t = 5$ the optimum for both sequences (with $NPV_5^{1,1} = 15,55$ and $NPV_5^{2,1} = 16,00$).

[10] The reader may assume that the initial term structure has the property that r_t for $t \to \infty$ is strictly monotonically decreasing towards $\bar{r} = 4,65\%$. Eventually future spot rates will be close to this value, too.

first sequence will have higher *NPVs* in the long run when a nearly flat term structure with interest rate around 4,65% prevails. When the term structure is "almost flat", Proposition 1 applies, i.e., the *NPVs* will decrease, and $NPV_t^{1,1} \geq NPV_t^{2,1}$ for large t according to Proposition 5.

Table 7: Term structures (inverse in $t = 0$)

time to maturity	1	2	3	4
$t = 0$	11,00%	10,00%	9,00%	8,00%
$t = 1$	9,00901%	8,01355%	7,01813%	6,02273%
$t = 2$	7,02719%	6,03636%	5,04559%	3,90767%
$t = 3$	5,05471%	4,06860%	2,88817%	2,94895%
$t = 4$	3,09174%	1,82171%	2,25645%	1,88898%
$t = 5$	0,56732%	1,84135%	1,49119%	2,11350%
$t = 6$	3,13153%	1,95630%	2,63416%	2,92545%
$t = 7$	0,79447%	2,38637%	2,85685%	3,28744%
$t = 8$	4,00342%	3,90381%	4,13205%	4,24138%
$t = 9$	3,80428%	4,19643%	4,32082%	4,37809%
$t = 10$	4,59005%	4,58006%	4,57006%	4,56007%
$t = 11$	4,57006%	4,56007%	4,55007%	4,57717%
$t = 12$	4,55007%	4,54008%	4,57954%	4,59922%
$t = 13$	4,53009%	4,59427%	4,61561%	4,62623%
$t = 14$	4,65850%	4,65840%	4,65830%	4,65820%
$t = 15$	4,65830%	4,65820%	4,65810%	4,65800%
$t = 16$	4,65810%	4,65800%	4,65790%	4,65780%
$t = 17$	4,65790%	4,65780%	4,65770%	4,65760%
$t = 18$	4,65770%	4,65760%	4,65750%	4,65740%
$t = 19$	4,65750%	4,65740%	4,65730%	4,65720%
$t = 20$	4,65730%	4,65720%	4,65710%	4,65700%

Fourthly, and most importantly, the highest *NPV* is achieved for the second sequence starting in $t = 5$, $NPV_5^{2,1}$. This contradicts the "general law" even when replacement and start are jointly optimized.

The *NPVs* of both sequences, absolute and relative to each other, depend on future spot rates, which are assumed to be equal to the forward rates implicit in the term structure of $t = 0$ (cf. Table 7). Theses numbers enable us to understand why e.g., $NPV_5^{2,1} > NPV_5^{1,1}$. Comparing sequences 1 and 2 (cf. Table 8) for a start in some t, it is clear that sequence 2 benefits from relatively small interest rates $r_{t,t+1}$ and $r_{t+2,t+3}$ and a relatively large interest rate $r_{t+1,t+2}$. [11] This qualitative statement, of course, has to be supplemented by noting that the absolute level of interest rates matters for the optimal starting date.

[11] This is indeed the case in $t = 5$: $r_{5,6} = 0,56732\%, r_{6,7} = 1,84135\%$, and $r_{7,8} = 1,49119\%$.

Table 8: Comparison of two sequences

	Cash Flows	IRR	NPV for $\bar{r} = 4,65\%$	NPV discounted to $t = 11$
Sequence 1,1	$-100; 13; 113$	13%	15,6034	9,4246
Sequence 2,1	$-100; 113; -101; 113$	12,50%	14,3515	8,6684

Let us now turn to the joint optimization of replacement and starting date for a normal term structure. There the derived contradiction to the "general law" already follows from Example 4, as can be seen when we apply the following extended term structure:

Table 9: Term structures (normal in $t = 0$)

time to maturity	1	2	3	4	5	6
$t = 0$	2,00%	2,50%	3,00%	3,50%	4,00%	4,50%
$t = 1$	3,00245%	3,50367%	4,00489%	4,50610%	5,00731%	5,50851%
$t = 2$	4,00733%	4,50976%	5,01218%	5,51459%	6,01699%	6,51939%
$t = 3$	5,01461%	5,51823%	6,02184%	6,52544%	7,02903%	7,39048%
$t = 4$	6,02427%	6,52908%	7,03387%	7,53865%	7,86001%	7,61547%
$t = 5$	7,03629%	7,54227%	8,04823%	8,32389%	7,93656%	7,62152%
$t = 6$	8,05064%	8,55778%	8,75652%	8,16281%	7,73895%	7,46068%
$t = 7$	9,06731%	9,11119%	8,20023%	7,66117%	7,34307%	7,13490%
$t = 8$	9,15508%	7,76928%	7,19650%	6,91629%	6,75255%	6,64688%

The cash flows of the machines are such that the only reasonable times of use are 1 and 3 (cf. Table 3). Therefore four cases have to be compared. Their total $NPVs$ are as follows:

Table 10: Example 3 with an extended normal term structure

t	0	1	2	3	4	5	6	7	8
$NPV_t^{1,1}$	14,50	12,14	9,83	7,61	5,54	3,66	1,99	1,11	2,52
$NPV_t^{1,3}$	18,82	13,73	8,98	4,66	0,85	-0,80	2,44	2,07	3,40
$NPV_t^{3,1}$	**19,49**	14,25	9,34	4,88	0,92	-1,91	-1,05	3,14	3,42
$NPV_t^{3,3}$	18,64	11,30	4,65	0,42	1,37	-0,95	-0,17	3,94	4,15

Apparently, we have another counterexample to the "general law", because $NPV_0^{3,1}$ is the maximum. This time, the term structure is normal and the optimum starting date equal to zero. Therefore we have shown:

Proposition 7

If the term structure is non-flat and the starting date of an investment chain as well as the optimal single identical replacement are jointly optimized, the optimal economic life of the first machine maybe longer than that of the second.

6 Conclusions

There are two well established facts in the investment literature which hold when decisions according to the *NPV* rule are made on the basis of a flat term structure:

- If a project has a positive *NPV* then it is better to start it immediately than to wait for one or more periods.
- In the case of a single identical replacement, the optimal economic life of the first machine is less than or equal to the optimal economic life of the second machine ("general law of replacement").

Both results seize to hold when non-flat term structures are allowed for. The paper shows that for inverse term structures the results do not carry on even for very simple projects whereas for a normal term structure more intriguing cases can be constructed.

These "counterexamples" to the general law do not only exist if the start of the investment chain is fixed, but also when it is optimized simultaneously.

We have ignored the possibility that the termination of the use of a machine involves extra sales revenues or scrapping costs which are not contained in the regular cash flow. Allowing for such complications would make the examples slightly more cumbersome, but would not yield qualitatively different results. Our observations obviously would continue to hold for small modifications of the sequences of cash flows directly, whereas for larger changes we would have to construct numerically different examples.

As outlined in the introduction, the situation examined, namely that of a single identical replacement, is certainly not the most realistic one. Still, the results achieved yield insights into an important benchmark case. In a broader perspective, they show that assuming flat term structures, often implicitly, is not as innocuous as it may have been thought.

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