5. Partial Autocorrelation Function

5.1 Definition, Computation, Estimation

Up to now:

- Ordinary autocorrelation function (ACF) of a stationary process \( \{X_t\}_{t \in \mathbb{Z}} \) at lag \( h \)
  \[ \rho_X(h) = \text{Corr}(X_t, X_{t-h}) \]
  measures the linear dependency among the process variables \( X_t \) and \( X_{t-h} \)

- However, the dependency structure among the intermediate process variables \( X_s, t-h < s < t \), plays an important role
Example: (I)

- Consider the stationary AR(1) process

\[ X_t = \phi X_{t-1} + \epsilon_t, \]

where \(|\phi| < 1\) and \(\epsilon_t \sim \text{WN}(0, \sigma^2)\)

- From Theorem 3.6 (Slides 74, 75) we know that

\[ \rho_X(h) = \phi^h \quad \text{for } h \geq 0, \]

and thus,

\[ \rho_X(2) = \text{Corr}(X_t, X_{t-2}) = \phi^2 > 0 \]

\( \rightarrow \) \(X_t\) and \(X_{t-2}\) are correlated
Example: (II)

- $X_t$ and $X_{t-2}$ are not directly correlated

- Actually, the correlation coefficient $\text{Corr}(X_t, X_{t-2}) = \phi^2$ results indirectly, since $X_t$ is correlated with $X_{t-1}$ and $X_{t-1}$ is correlated with $X_{t-2}$

Obviously:

- The ordinary ACF comprises all (direct and indirect) correlation between $X_t$ and $X_{t-h}$

Intuitive idea:

- Consider only the direct correlation between $X_t$ and $X_{t-h}$
  $\longrightarrow$ partial autocorrelation function (PACF)
Definition 5.1: (Partial autocorrelation function (PACF))

Let \( \{X_t\}_{t \in \mathbb{Z}} \) be a stationary process. The partial autocorrelation at lag \( h \) for \( h \geq 2 \) [in symbols: \( \pi_X(h) \)], is defined as the direct correlation between \( X_t \) and \( X_{t-h} \) with the linear dependence between the intermediate variables \( X_s \) with \( t - h < s < t \) removed. For the sake of completeness we set

\[
\begin{align*}
\pi_X(0) &= 1, \\
\pi_X(1) &= \rho_X(1), \\
\pi_X(h) &= \pi_X(-h) \quad \text{for} \ h < 0.
\end{align*}
\]

\[ \blacksquare \]
Question:

- How can we compute the PACF of a stochastic process \( \{X_t\}_{t \in \mathbb{Z}} \)?

**Theorem 5.2: (Computation of the theoretical PACF)**

Let \( \{X_t\}_{t \in \mathbb{Z}} \) be a stationary process. The partial autocorrelation \( \pi_X(h) \) for \( h \geq 2 \) is equal to the coefficient \( \alpha_h \) from the optimal linear prediction of \( X_t \) on the basis of the observations \( X_{t-1}, \ldots, X_{t-h} \):

\[
\hat{X}_{t-1,1} = \alpha_1 X_{t-1} + \cdots + \alpha_h X_{t-h}.
\]

(Proof: Schlittgen & Streitberg, 2001)
Remark:

- There is a specific mathematical procedure for computing the coefficients $\alpha_1, \ldots, \alpha_h$ given in Theorem 5.2 (Levinson-Durbin recursion)

  $\rightarrow$ theoretical partial autocorrelation $\pi_X(h) = \alpha_h$

Now:

- Estimation of the PACF on the basis of a sample $X_1, \ldots, X_T$ (cf. Chapter 4)
Estimation procedure: (I)

- Let $X_1, \ldots, X_T$ be a sample (trajectory) from the theoretical process $\{X_t\}_{t \in \mathbb{Z}}$

- We consider the following regression equation:

$$X_t = \beta_0 + \beta_1 X_{t-1} + \ldots + \beta_h X_{t-h} + u_t$$

($u_t$ is a classical error term)

- We estimate the unknown parameters $\beta_0, \ldots, \beta_h$ by OLS (ordinary least squares estimators)
Estimation procedure: (II)

- We estimate $\pi_X(h)$ by the OLS estimator $\hat{\beta}_h$ of the above-stated regression equation

$$\hat{\pi}_X(h) = \hat{\beta}_h$$

- Approximatively, we have

$$\text{Var}[\hat{\pi}_X(h)] \approx \frac{1}{T}$$
Remarks:

- In general, there are no analytically closed-form formulae available for estimating the PACF $\pi_X(h), h \geq 0,$ of an arbitrary $\text{ARMA}(p, q)$ process.

- The theoretical ACFs and PACFs of $\text{ARMA}(p, q)$ processes exhibit the following pattern:
<table>
<thead>
<tr>
<th>Prozess</th>
<th>ACF ($\rho_X(h)$)</th>
<th>PACF ($\pi_X(h)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR($p$)</td>
<td>infinite (dampened exponential or sinusoidal waves)</td>
<td>finite $\pi_X(h) = 0$ for $h &gt; p$</td>
</tr>
<tr>
<td>MA($q$)</td>
<td>finite $\rho_X(h) = 0$ for $h &gt; q$</td>
<td>infinite (dampened exponential or sinusoidal waves)</td>
</tr>
<tr>
<td>ARMA($p, q$)</td>
<td>as AR($p$) for $h &gt; q$</td>
<td>as MA($q$) for $h &gt; p$</td>
</tr>
</tbody>
</table>
5.2 Interpretation of ACF and PACF

Remarks:

• We try to use the estimated ACF and PACF of an observed time series to identify the underlying unknown data-generating process

• We compare the estimated ACFs and PACFs with the patterns of their theoretical counterparts (cf. the table on Slide 176)

• In the case of ARMA($p,q$) models of low orders (e.g. for $p+q \leq 3$) this identification strategy often yields good results (cf. Stralkowski et al., 1974)

• For mixed ARMA($p,q$) models of higher orders the identification strategy is often less successful
Examples:

- For $\epsilon_t \sim \text{GWN}(0, 1)$ we consider the estimated ACFs and PACFs of the processes

  - $X_t = \epsilon_t - 0.8\epsilon_{t-1}$ \hspace{2cm} \text{MA(1)}
  
  - $X_t = 0.8X_{t-1} + \epsilon_t$ \hspace{2cm} \text{AR(1)}
  
  - $X_t = 1.3X_{t-1} - 0.4X_{t-2} + \epsilon_t + 0.4\epsilon_{t-1}$ \hspace{2cm} \text{ARMA(2, 1)}

(cf. Slides 59, 78, 123)
Estimated ACF of an MA(1) process

Estimated PACF of an MA(1) process
Estimated ACF of an AR(1) process

Estimated PACF of an AR(1) process
Estimated ACF of an ARMA(2,1) process

Estimated PACF of an ARMA(2,1) process