SDF MODEL AND THE C-CAPM

In the one-period standard-CAPM, the investor’s objective function is assumed to be fully determined by the (one-period) standard deviation and expected return on the portfolio.

Equilibrium returns then arise as a consequence that all agents have the same expectations and all assets must be willingly held.

An alternative view of the determination of equilibrium returns is provided by the C-CAPM.

Here, the investor maximises expected utility that depends only on current and future consumption (see Lucas 1978, Mankiw and Shapiro 1986, Cochrane 2001).

Financial assets play a role in this model in that they help to smooth consumption over time.
Securities are held to transfer purchasing power from one period to another. If she holds assets, she can sell some of these to finance consumption when her current income is low.

An individual asset is therefore more ‘desirable’ if its return is expected to be high when consumption is expected to be low.

Thus, the systematic risk of the asset is determined by the covariance of the asset’s return with respect to consumption (rather than its covariance with respect to the return on the market portfolio as in the ‘standard’ CAPM).

A formal derivation (Cochrane 2001). Consider the two-period case for simplicity, with a *time-separable* utility

$$\max U = U(C_t) + \theta E_t [U(C_{t+1})]$$

(2)

The constraints are

$$C_t = \tilde{C}_t - P_t N$$

(3a)

$$C_{t+1} = \tilde{C}_{t+1} + X_{t+1} N$$

(3b)
where \( C_t = \) real consumption

\( \tilde{C}_t = \) consumption levels before purchase of the asset

\( P_t = \) price of (any) asset

\( N = \) number of units of asset purchased

\( X_{t+1} \equiv P_{t+1} + D_{t+1} = \) pay-off at \( t + 1 \)

\( \theta = \) subjective discount factor \((0 < \theta < 1)\)

Time separability implies that marginal utility in any period does not depend on consumption in other periods.

Using the constraints (3a) and (3b) and maximising (2) with respect to \( N \)

\[
P_t U'(C_t) = E_t \left[ \theta U'(C_{t+1})X_{t+1} \right]
\]

which can be rearranged to give the key pricing relationship for any asset:

\[
P_t = E_t(M_{t+1}X_{t+1})
\]
where for the C-CAPM, the *stochastic discount factor* is

\[ M_{t+1} = \frac{\theta U'(C_{t+1})}{U'(C_t)} \]  

\[(6)\]

\( M_{t+1} \) is also referred to as the *pricing kernel* and in the C-CAPM it is the *marginal rate of substitution* (MRS) between current and future consumption.

From (5), we deduce that returns are determined as

\[ 1 = E_t [M_{t+1}R_{t+1}^*] \]  

\[(10)\]

If a risk-free security is traded, then we can take ‘\( R \)’ outside of the expectation in (10) so that

\[ R_{ft}^* = \frac{1}{E_t(M_{t+1})} \]  

\[(11)\]
**Power utility and lognormal consumption growth**

With power utility, we have

\[
U(C_t) = \frac{1}{(1-\gamma)} C_t^{1-\gamma}
\]

and note that as \( \gamma \to 1 \) we have ‘log utility’, \( U(C_t) = \ln(C_t) \). From (6) and (12),

\[
M_{t+1} = \theta(C_{t+1}/C_t)^{-\gamma}
\]

A standard statistical result is that if \( Z \) is lognormal, then \( \ln Z \sim N(\mu_{\ln Z}, \sigma^2_{\ln Z}) \) and

\[
E[kZ] = \exp[k\mu_{\ln Z} + (1/2)k^2\sigma^2_{\ln Z}]
\]

where \( k \) is a constant. Using (11), (13) and (14), the risk-free rate is given by

\[
R^*_f = \left[ \theta \left\{ \exp \left[ -\gamma \mathbb{E}_t \Delta c_{t+1} + \frac{\gamma^2}{2} \sigma_t^2(\Delta c_{t+1}) \right] \right\} \right]^{-1}
\]
where $\Delta c_{t+1} \equiv \ln(C_{t+1}/C_t)$. Taking logarithms

$$\ln R^*_t = -\ln \theta + \gamma E_t \Delta c_{t+1} - \frac{\gamma^2}{2} \sigma_t^2(\Delta c_{t+1})$$  \hspace{1cm} (16)$$

Note that $\ln R^*_t = \ln(1+R_{ft}) \approx R_{ft}$.

From (16), we can deduce the following relationship between consumption growth and the risk-free rate:

(a) Real interest rates are high when expected consumption growth is high. High real interest rates are required to lower consumption today in order to save today and then increase consumption tomorrow.

(b) Real interest rates are high when $\theta$ is low (for given $E_t \Delta c_{t+1}$).

(c) With higher volatility of consumption, people want to save more and real interest rates are driven lower. This is a form of precautionary saving.
(d) If $\gamma = 0$ (i.e. linear utility function and hence no risk aversion), then the real rate is constant and equal to the subjective discount factor, $\theta$.

(e) As the curvature $\gamma$ of the (power) utility function increases, the real rate is more responsive to consumption growth.

Note that in equation (16), consumption is endogenous, so we could also interpret our ‘correlations’ in the opposite direction.

Returns on Risky Assets

What determines the movement in asset returns over time and what determines the average return on risky asset $i$, relative to that on risky asset $j$?

The key determinant of both these effects is the size of the covariance between the stochastic discount factor $M_{t+1}$ and the asset return.

The C-CAPM in terms of returns for any risky asset $i$ is (where we drop some time subscripts for notational ease):
\[ 1 = E_t (MR^*_i) \]  

(17)

For any two random variables \( x \) and \( y \),

\[ E(xy) = \text{cov}(x, y) + (Ex)(Ey) \]  

(18)

hence using (17) and (18):

\[ 1 = E_t(M)E_t(R^*_i) + \text{cov}_t(M,R^*_i) \]  

(19)

\[ E_t(R^*_i) = \left[ 1 - \text{cov}_t(R^*_i, M) \right] / E_t(M) \]  

(20)

Using \( R^*_f = 1/E_t(M) = U'(C_t)/\theta E_tU'(C_{t+1}) \) in (20) and noting that

\[ \text{cov}_t(M,R^*_i) = \theta \text{cov}_t[U'(C_{t+1}),R^*_i]/U(C_t), \]

we obtain a key equation that determines the excess return on any risky asset \( i \):

\[ (E_tR^*_i - R^*_f)_{t+1} = -R^*_f \text{cov}_t(M,R^*_i) = -\text{cov}_t[U'(C_{t+1}),R^*_{ist+1}] / E_t[U'(C_{t+1})] \]  

(21)
The relative (expected excess) return of two assets $i$ and $j$ differ only because the covariance of $R_i$ or $R_j$ with (the marginal utility of) consumption is different.

An asset whose return has a negative covariance with $U'(C_{t+1})$ and hence a positive covariance with $C_{t+1}$ will have to offer a high expected return, in order that investors are willing to hold the asset.

This is because the asset pays off when consumption is high, and consumers are already ‘feeling good’ and so the higher return gives them little additional utility.

Conversely, asset returns that co-vary negatively with consumption (i.e. positively with $U'(C_{t+1})$ provide insurance and will be willingly held, even if they promise a low expected return.

If we assume joint lognormality of consumption growth and asset returns, it can be shown that excess returns are given by

$$E_t(r_{i,t+1}^*-r_{f,t}^*) + (1/2)\sigma_t^2 (r_{i,t+1}^*) = -\text{cov}_t(m_{t+1}, r_{i,t+1}^*)$$
where \( r_{i,t+1}^* = \ln R_{i,t+1}^* \), and so on.

The second term is the Jensen effect and the covariance term is the risk premium.

**Prices and Covariance**

Using (18) in \( P_t = E_t(MX) \) and substituting \( R_f^* = 1/E(M) \)

\[
P_t = \frac{E_t(X_{t+1})}{R_f^*} + \text{cov}_t(M_{t+1}, X_{t+1})
\]

(27a)

which after incorporating the definition of \( M_{t+1} \) gives

\[
P_t = \frac{E_t(X_{t+1})}{R_f} + \text{cov}_t[\theta U'(C_{t+1}), X_{t+1}] / U'(C_t)
\]

(27b)
If there is no risk aversion (i.e. utility is linear in consumption) or if consumption is constant, then the usual ‘risk-neutral’ formula applies:

\[ P_t = \frac{E_t(X_{t+1})}{R^*_f} \]  \hspace{1cm} (28)

So under risk-neutrality, the price of a risky asset is the expected present value of its pay-off \( X_{t+1} \) discounted using the \textit{risk-free} rate.

Under risk aversion, (27) applies, and the second term in (27a) or (27b) is a \textit{risk adjustment}.

Rational Valuation Formula and SDF

We can derive the Rational Valuation Formula for any asset, using the SDF equilibrium condition

\[ E_t(R^*_{t+1}M_{t+1}) = 1 \]  \hspace{1cm} (29)

and the definition
\[ R^*_{t+1} \equiv \frac{(P_{t+1} + D_{t+1})}{P_t} \] (30)

We then obtain

\[ P_t = E_t(M_{t+1}[P_{t+1} + D_{t+1}]) \] (31)

Repeated forward substitution then yields (assuming the transversality condition holds)

\[ P_t = E_t \sum_{j=1}^{\infty} M_{t,t+j} D_{t+j} \quad \text{where} \quad M_{t,t+j} = M_{t,1} M_{t,2} \cdots M_{t,j} \] (32)

\( M_{t,t+j} \) is a possibly time-varying stochastic discount factor.

Using (18), we can replace the \( E(xy) \) term in (32) to give

\[ P_t = \sum_{j=1}^{\infty} \frac{E_t D_{t+j}}{R^*_{j,t+j}} + \sum_{j=1}^{\infty} \text{cov}_t(D_{t+j} M_{t,t+j}) \] (33)

where \( 1/R^*_{f,t+j} \equiv E_t(M_{t,t+j}) \) is the \( j \)-period risk-free interest rate.
Should Returns be Predictable in the C-CAPM?

Is the stochastic discount factor (SDF) model consistent with the stylised fact that returns over short horizons (e.g. intraday, over one day, or one week) are virtually unpredictable and hence price follows a martingale process?

Remember that $P_t$ is a *martingale*, if $P_t = E_tP_{t+1}$ or $P_{t+1} = P_t + \varepsilon_{t+1}$  \hspace{1cm} (1)

where $E_t\varepsilon_{t+1} = 0$ and $\varepsilon_{t+1}$ is independent of $P_t$.

For a martingale, the return (with zero dividends) is $E_tP_{t+1}/P_t = 1$, a constant, and if in addition $\varepsilon_t$ is *iid*, then prices follow a *random walk*.

The first-order condition for the SDF model is

$$P_tU'(C_t) = E_t [\theta'(C_{t+1})(P_{t+1} + D_{t+1})]$$  \hspace{1cm} (2)

Hence, the SDF model implies $P$ is a martingale if
(a) investors are risk-neutral (i.e. $U(C)$ is linear so $U'(C) = \text{constant}$) and
(b) no dividends are paid between $t$ and $t+1$ and
(c) $\theta$ is close to 1

(a)–(c) are not unreasonable over short horizons.

Of course, if daily price changes are definitely unpredictable, this would invalidate ‘technical analysis’ (e.g. chartism, candlesticks, neural networks) as a method of making money (corrected for risk and transactions costs).

Longer Horizons

Long-horizon stock returns (e.g. over one- to five-year horizons) appear to exhibit some predictability (although the relationships uncovered are not necessarily stable over different time periods).

Is the C-CAPM consistent with this stylized fact?

In terms of returns, the C-CAPM for any risky-asset return $R$ gives
\[ E_t R_{t+1}^* - R_{f,t}^* = \frac{-\text{cov}_t(M_{t+1}, R_{t+1}^*)}{E_t(M_{t+1})} = -\left[ \frac{\sigma_t(M_{t+1})}{E_t(M_{t+1})} \right] \sigma_t(R_{t+1}^*) \rho_t(M_{t+1}, R_{t+1}^*) \] (3)

where \( R_{t+1}^* = (1 + R_{t+1}) \) and so on.

If we now assume (for simplicity) power utility and lognormal consumption growth, then

\[ E_t R_{t+1}^* - R_{f,t}^* \approx [\gamma_t \sigma_t(\Delta c_{t+1})] \sigma_t(R_{t+1}^*) \rho_t(M_{t+1}, R_{t+1}^*) \] (4)

Key contenders for explaining changes in equilibrium excess returns over time include either time-varying risk aversion \( \gamma_t \) or time-varying volatility of consumption.

Hansen–Jagannathan Bounds and the Sharpe Ratio

Using \( \text{cov}(R_i^*, M) \equiv \rho_{iM} \sigma(R_i^*) \sigma(M) \) and substituting in \( 1 = E(MR_i^*) \),

\[ 1 = E(MR_i^*) = E(M)E(R_i^*) + \rho_{iM} \sigma(R_i^*) \sigma(M) \] (5)
Rearranging and using $E(M) = 1/R_f$,

$$(ER^*_i - R^*_f) / \sigma(R^*_i) \equiv E(R^*_e_i) / \sigma(R^*_e_i) = -\rho_{iM}(\sigma(M) / E(M))$$

(6)

where $R^*_e_i$ is the excess return.

The term on the left is the Sharpe ratio for an asset or portfolio of risky assets.

For any portfolio $i$, equation (6) gives the Hansen and Jagannathan (1991) bound for the discount factor.

Since $\rho_{iM}$ has an absolute maximum value of 1,

$$\sigma(M) / E(M) \geq |E(R^*_e_i)| / \sigma(R^*_e_i)$$

(7)

Given that the right-hand side is measurable, then equation (7) provides a lower bound for the behaviour of the SDF for any asset $i$. 
We can now connect our SDF approach with the standard-CAPM, mean-variance model.

First, all asset returns lie within the wedge-shaped region of Figure 1.
By combining risky assets into a portfolio with minimum variance for any given level of expected return, we obtain mean-variance efficient portfolios (when we have a riskless asset).

All returns on the wedge-shaped frontier are perfectly correlated with the SDF, M so that $|\rho_{iM}| = 1$.

Returns on the upper portion (i.e. equivalent of the CML) are perfectly negatively correlated with M and hence positively correlated with consumption and from (6) command the highest expected return.

Conversely, those assets (or portfolios) on the lower portion of the wedge have $\rho_{iM} = +1$ and hence have lower expected returns because $R_i$ and consumption are perfectly negatively correlated, and these assets provide insurance against changes in consumption.

Any two returns on the wedge-shaped mean-variance frontier ($R_{mv,1}, R_{mv,2}$) are perfectly correlated with each other (because each is perfectly correlated with M)
The expected excess return on any asset $i$ is proportional to its beta with any return on the efficient frontier $\beta_{i,mv}$

$$ER_i - R_f = \beta_{i,mv}(ER_{mv} - R_f)$$

and the factor risk premium $\lambda_M = ER_{mv} - R_f$.

In facts, from (3), for any asset $i$, we have

$$E_t R_{i,t+1} - R_f = -\frac{\text{cov}_t(M_{t+1}, R_{i,t+1})}{E_t(M_{t+1})}$$

$$\frac{\text{cov}_t(M_{t+1}, R_{i,t+1})}{\text{var}_t(M_{t+1})} [-\frac{\text{var}_t(M_{t+1})}{E_t(M_{t+1})}] = \beta_{i,M} \lambda_M$$

where $\beta_{i,M}$ is the coefficient of $R_i$ regressed on $M$, while $\lambda_M$ is independent of any asset $i$ and can be interpreted as the market price of risk.

For power utility it can be shown (by taking a Taylor series expansion of the above equation):

$$E_t R_{i,t+1} - R_f \approx \beta_{i,\Delta C_{t+1}} \gamma \text{var}_t(\Delta C_{t+1})$$

So the SDF model with power utility implies that the market price of risk depends
positively on $\gamma$ and the riskiness of consumption.

The direct parallels between the standard-CAPM (mean-variance model) and the SDF model should be apparent in the above equations.

all portfolios on the efficient frontier have the same Sharpe ratio and the ‘fundamentals’ that determine the size of the Sharpe ratio are the risk-free rate and the volatility of the SDF.

Since both of these variables may move over time, we also expect the measured Sharpe ratio to vary.

Using power utility and assuming consumption growth is lognormal,

$$\frac{\sigma(M)}{E(M)} = \sigma\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right\} / E\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right\} =$$

$$= \left[\exp\left\{\gamma^2 \sigma^2(\Delta c_{t+1})\right\} - 1\right]^{1/2} \approx \gamma \sigma(\Delta c)$$

(9a)
Hence, from (8) and (9a), a high observed Sharpe ratio for assets or portfolios on the efficient frontier is consistent with the C-CAPM if consumption growth is volatile or $\gamma$ is large.

Both of these are indicators of ‘riskiness’ in the economy.

Also, the Sharpe ratio moves over time with the changing conditional volatility of consumption growth.

For any asset $i$ that is not on the efficient frontier (CML), the Sharpe ratio (from (6)) under power utility and lognormality should equal

$$Sr_i \approx -\rho_i M \gamma \sigma(\Delta c)$$