INTERTEMPORAL ASSET ALLOCATION: THEORY

Multi-Period Model
The agent acts as a price-taker in asset markets and then chooses today’s consumption and asset shares to maximise *lifetime* utility.

This multi-period problem can be transformed into a sequence of two-period problems by invoking the concept of a *value function*.

The value function is a recursive relationship that can be ‘solved backwards’ from the terminal date and is often referred to as the *Bellman equation* in the stochastic dynamic programming literature.

So, at $t$, to calculate today’s optimal values, you have to calculate all the optimal values ‘backwards’ from $T$ to $t$. 
Preferences and Budget Constraint

The individual is assumed to maximise a time-separable utility function that depends (only) on current consumption $C_o$ and all future consumption $(C_1, C_2, ...)$ and the current period is $t = 0$

$$ U = E_0 \sum_{t=0}^{T-1} U(C_t) + B(W_T) $$

where $B(.)$ is the bequest function at time $T$.

For power utility, we have

$$ U(C_t) = \theta^t C_t^{1-\gamma}/(1 - \gamma) $$

$$ B(W_T) = \theta^T W_T^{1-\gamma}/(1 - \gamma) $$

The budget constraint is

$$ W_{t+1} = R_{t+1}(W_t - C_t) $$
and the *portfolio* return is

\[
R_{t+1} = \sum_{i=1}^{m} \alpha_{it}(R_{i,t+1} - R_{ft}) + R_{ft}
\]  

(31)

with \( \alpha_{it} \) being the asset shares to be determined and \( R_i \) are the \( m = n - 1 \) risky asset returns.

The risk-free asset share is given by \( \alpha_n = 1 - \sum_{i=1}^{m} \alpha_i \).

The general closed-form solution at \( t \) to the intertemporal problem will be of the form

\[
C_t = f(W_t, E_{tR_{t+j}}, \gamma, \theta, T - t)
\]  

(27a)

\[
\alpha_{it} = g(W_t, E_{tR_{t+j}}, \gamma, \theta, T - t)
\]  

(27b)

where \( \gamma \) represents parameter(s) from an intertemporal utility function.

Closed-form solutions are often not possible, but for power and logarithmic (\( \gamma = 1 \)) time-separable utility functions, we can make some progress in this direction.
The Value Function

To solve these multi-period problems, we introduce the value function $J(W_t)$ (sometimes called the derived utility of wealth function) defined as

$$J(W_t) = \max E_t\{\sum_{s=t}^{T-1} U(C_s) + B(W_T)\} \quad (32)$$

The value function will allow us to express the multi-period problem as a sequence of two-period problems.

It follows that

$$J(W_T) = B(W_T) \quad (33)$$

At $T - 1$, the individual chooses $C_{T-1}, \alpha_{T-1}$ to maximise

$$J(W_{T-1}) = \max_{C_{T-1}, \alpha_{T-1}} \{U(C_{T-1}) + E_{T-1}B(W_T)\} \quad (34)$$

Substituting $W_{t+1} = R_{t+1}(W_t - C_t)$ and noting from (31) that $R_{t+1}$ is a function of $\alpha_{it}$,
(34) becomes

\[ J(W_{T-1}) = \max_{C_{T-1}, \alpha_{T-1}} \{ U(C_{T-1}) + E_{T-1}\{B(R_T (\alpha_{i,T-1})(W_{T-1} - C_{T-1}))\} \} \quad (35) \]

and we have a familiar two-period problem.

The FOC for \( C_{T-1} \) gives

\[
0 = U'(C_{T-1}) + E_{T-1}\{B(W_T)(\partial W_T / \partial C_{T-1})\} \quad (36)
\]

or

\[
0 = U'(C_{T-1}) - E_{T-1}\{B(W_T)\left[ \sum_{i=1}^{m} \alpha_{i,T-1}(R_{i,T} - R_f) + R_f \right]\} \quad (38b)
\]

The FOC for \( \alpha_i \) after differentiating (34) and using (30) and (31) is

\[
0 = E_{T-1}\{B(W_T)(R_{i,T} - R_f)\} \quad (i = 1, 2, 3, \ldots, m) \quad (39)
\]
In principle, the single FOC (38b) and the \( m \) FOCs (39) are \( m + 1 \) equations in \( m + 1 \) unknowns, namely, \( C_{T-1} \) and the \( m \) values \( \alpha_{i,T-1} \).

These can be solved for the optimal asset shares and consumption at \( T - 1 \).

As in the two-period problem one can show that, in the case of power utility, \( \alpha^*_i \) are independent of wealth – therefore, all investors hold the same asset proportions.

In general, \( \alpha_{i,T-1} \) would also be a function of wealth.

Asset proportions also depend on expected returns but if returns are \( iid \), then \( \alpha_{i,t} \) is the same regardless of the planning horizon of the investor, \( T - t \).

This special case is often referred to as myopic behaviour, because in a multi-period model, optimal asset proportions are identical to those of a one-period investor.
Recursions Unlimited

Since we have optimal values for $C^*_{T-1}$ and $\alpha^*_{i,T-1}$, we have an optimal value for $J(W_{T-1})$ from (34).

Using the value function (34), we can set out the optimisation problem for $T - 2$

$$J(W_{T-2}) = \max_{C_{T-2}, \alpha_{T-2}} E_{T-2}[U(C_{T-2}) + U(C^*_{T-1}) + B(W_T)]$$

(49)

At $T - 2$, $W_{T-2}$ is known, and the principle of optimality states that at $T - 2$, we can choose $C_{T-2}, \alpha_{i,T-2}$ and given this outcome at $T - 2$, the remaining decisions for $T - 1$ must be optimal.

Furthermore, using iterated expectations, we can write (49) as

$$J(W_{T-2}) = \max \left\{ U(C_{T-2}) + E_{T-2} \left\{ \max_{C_{T-1}, \alpha_{T-1}} E_{T-1}[U(C^*_{T-1}) + B(W_T)] \right\} \right\}$$

(50)
\[ J(W_{T-2}) = \max \left\{ U(C_{T-2}) + E_{T-2}[J(W_{T-1})] \right\} \]  

(51)

where \( J(W_{T-1}) \) is calculated using the optimal values \( C^*_{T-1}, W^*_i, T-1 \) found at \( T - 1 \).

Intuitively, (51) has the utility function dependent on this period’s consumption \( C_{T-2} \) and next period’s wealth \( W_{T-1} \).

If I consume more at \( T - 2 \), then wealth next period will be lower and hence there is a trade-off.

Equation (51) is a recursion for \( J(.) \), and the solution to (51) will be of the same form as that for (34), which we have already noted.

Hence, in general, for any \( t = 0, 1, 2, \ldots, T - 1 \), the optimality conditions by analogy with (42) and (39) with \( J_W \) replacing \( B_W \) are

\[ U'(C_t) = R_{ft} E_t J_W(W_{t+1}) = J_W(W_t) \]  

(52a)
\[ E_t\{R_{it+1}J_W(W_{t+1})\} = R_{it}E_t\{J_W(W_{t+1})\} \text{ for } i = 1, 2, 3, \ldots, m, \text{ assets (52b)} \]

This representation of the FOC for consumption implies that along the optimum path, a dollar saved that adds to wealth should give the same marginal utility as a dollar spent on consumption.

The FOCs in (52a) and (52b) look rather ‘neat’ and succinct, but a closed-form solution is generally only possible for a small subset of admissible utility functions.

In general, optimal asset proportions and consumption at any time \( t \) will both depend on wealth, expected returns from \( t \) to \( T-1 \) and the parameters of the utility function.

A numerical solution would give a single value for \( C_t \), given that \( W_t \) is a known input.

Hence, in more complex cases, we have a computable equilibrium model on which we can perform ‘what if’ numerical simulations.
Indeed, we can obtain the optimal consumption path $C_t, C_{t+1}, \ldots, C_T$ and the optimal asset proportions in the current and all future periods once we have the state variable $W_t$ and a ‘forecast’ of expected returns (in all future periods).

The ‘expectations’ must be calculated numerically.

In the above model, the only ‘exogenous’ stochastic variables are asset returns.

If we add ‘uncertain’ labour income into the budget constraint, we have to deal with any correlation between returns and labour income when calculating expectations – this considerably complicates the numerical solution procedure (Campbell and Vicera 1999).
INTERTEMPORAL ASSET ALLOCATION: EMPIRICS

Bodie, Merton and Samuelson (1992) include certain labour income (with retirement) in the intertemporal consumption-portfolio model (with power utility and iid returns).

Labour income is a non-tradeable asset (i.e. you cannot borrow against future labour income, and human capital is the value today of this future income).

In this model, certain labour income acts like a risk-free asset and ‘crowds out’ the latter, leading to an increased share held in the risky assets.

Hence, $\alpha^e > \alpha^r$, where $e = \text{‘in employment’}$ and $r = \text{‘in retirement’}$.

The certain labour income in employment implies that more will be invested in the risky asset (than in retirement, when labour income is assumed to be zero).

If labour income is uncertain but uncorrelated with risky returns, the tilt towards risky assets is less than in the certain income case (Viceira 2001).
Viceira (2001) addresses the portfolio-consumption problem when income is stochastic and may be *contemporaneously correlated* with risky-asset returns.

We now have an additional state variable, labour income, and this can give rise to a hedging demand for risky assets (first noted by Merton (1971) in an intertemporal model with time-varying expected returns).

Hedging demand arises from the desire to reduce lifetime consumption risk, and here this risk arises from the correlation between returns and income.

If shocks to income are negatively related to shocks to returns, then stocks are ‘desirable’ since they provide a return when income is low and hence help smooth out consumption.

In Viceira’s model, agents maximise expected intertemporal power utility subject to a budget constraint

$$\max_{\{c_t, \alpha_t\}_0^\infty} E \sum_{t=0}^\infty \theta^t U(C_t)$$  \hspace{1cm} (1a)
\[ W_{t+1} = (W_t + Y_t - C_t)R_{p,t+1} \]  \hspace{1cm} (1b)

\( 0 < \theta < 1 \) is the discount factor and \( Y_t \) is employment (labour) income, which is zero in the retirement state.

Labour income is uninsurable – you cannot write claims against your future income.

There are no labour supply decisions, so income is exogenous.

The portfolio return is

\[ R_{p,t+1} = \alpha_t(R_{1,t+1} - R_f) + R_f \]  \hspace{1cm} (2)

where \( R_{1,t+1} \) is the gross return on the \textit{single} risky asset, \( R_f \) is the gross risk-free rate and \( \alpha_t \) the risky-asset share.

The share held in the riskless asset is \((1 - \alpha_t)\).
The natural logarithm of income is a random walk with drift $g$, and expected excess (log) returns $(r_{1,t+1} - r_f)$ are assumed constant

$$Y_{t+1} = Y_t \exp(g + \varepsilon_t)$$  \hspace{2cm} (3a)

$$r_{1,t+1} - r_f = \mu + u_{t+1}$$  \hspace{2cm} (3b)

$$\text{var}_t(u_{t+1}) = \sigma_u^2, \quad \text{var}_t(Y_{t+1}) = \sigma_{\varepsilon}^2, \quad \text{cov}_t(u_{t+1}, \varepsilon_{t+1}) = \sigma_{\varepsilon u}$$  \hspace{2cm} (3c)

The error term $u_{t+1}$ is independent of the employment/retirement state.

In addition, consumption growth and asset returns are assumed jointly lognormal.

The employment–retirement state is random.

Employment occurs with a probability $\pi^e$ and retirement with probability $\pi^r = 1 - \pi^e$ ($0 < \pi^r < 1$), with retirement being irreversible, and labour income in retirement is zero.
After retirement, there is a constant probability of death $\pi^d$, so people live $1/\pi^d$ years after retirement, while the expected number of years to retirement is $1/(1 - \pi^e)$.

Because there is zero income in the retirement state (denoted by superscript $r$), we have our standard model with Euler equation

$$1 = E_t\{\theta^r (C^r_{t+1}/C^r_t)^{-\gamma}R_{i,t+1}\}$$

(4)

where $\theta^r = (1 - \pi^r)\theta$ and $R_{i,t+1}$ can be either $i = 1, f$ or $p$, that is, the risky asset, the risk-free asset or the portfolio return.

Viceira provides approximate solutions on the basis of a second-order Taylor series expansion, so that precautionary savings effects (i.e. volatility terms) are included (Campbell and Viceira 1999).

This provides some useful intuitive insights and allows a closed-form solution for both consumption and asset shares.

For the retirement state, optimal (log) consumption and portfolio shares are
\[ c^r_t = b^r_0 + w_t \quad \quad (5a) \]

\[ \alpha^r_t = (\mu + \sigma^2_u / 2) / (\gamma \sigma^2_u) \quad \quad (5b) \]

where \( b^r_0 \) is a value depending on \( \gamma, \theta \) and on the expected value and on the variance of \( r^r_{p,t+1} \).

For log utility investors, \((C/W)_t\) is independent of returns.

For more risk-averse (retired) investors (i.e. \( \gamma > 1 \)), the \((C/W)_t\) ratio is increasing in expected portfolio returns because the income effect of an increase in \( E r_p \) on wealth outweighs the substitution effect (i.e. save more today).

The impact of \( \text{var}_t (r_{p,t+1}) \) on \((C/W)_t\) is zero for \( \gamma = 1 \) but otherwise has a negative impact – the greater the uncertainty about \( r_p \), the lower is \((C/W)\) and the higher is today’s saving (i.e. precautionary savings).
Hedging Demand

What about consumption and asset shares in the employment state?

In the employment state, there is a probability you will stay employed and receive labour income and a probability you will enter retirement, hence the Euler equation is

$$ 1 = E_t \left\{ \left[ \pi^e \theta^e \left( \frac{C^e_{t+1}}{C^e_t} \right)^{-\gamma} + \left( 1 - \pi^e \right) \theta^r \left( \frac{C^r_{t+1}}{C^r_t} \right)^{-\gamma} \right] R_{it+1} \right\} $$

(6)

for $i = 1, f$ or $p$ and $\theta^e = \theta$, while $\theta^r = (1 - \pi^r) \theta$.

The solution for (log) consumption and the risky-asset share is

$$ c^e_t - y_t = b^e_0 + b^e_1 (w_t - y_t) $$

(7a)

$$ \alpha^e = \frac{(\mu + \sigma^2_u / 2)}{(\gamma b_1 \sigma^2_u) - (\pi^e (1 - b^e_1) \sigma_{eu}) / (b_1 \sigma^2_u) } $$

(7b)
where $0 < b^e_1 < 1$, $b_1 = \pi^e b^e_1 + (1 - \pi^e) < 1$ and $b^e_1$ is a constant that is an increasing function of $g$ (income growth) and also depends on the variance of income growth, the variance of portfolio returns and the covariance between these two.

The dependence of $\alpha^e$ on the expected number of years until retirement, that is, $(1 - \pi^e)^{-1}$, implies that asset shares (of the employed) do depend on the ‘horizon’ considered.

Consider the special case $\sigma_{eu} = 0$ so income shocks are uncorrelated with shocks to asset returns. Then it can be shown that $\alpha^e > \alpha^r$. (See $\alpha^r = (\mu + \sigma^2_u / 2) / (\gamma \sigma^2_u)$ (5b))

Also it can be shown that the longer are expected years to retirement, the larger is $\alpha^e$, even when returns are iid (see Viceira 2001, Table 1).

Now consider the hedging demand, that is the second term of (7b).

The ‘hedging term’ is $\beta = \sigma_{eu} / \sigma^2_u$, which is the regression slope of labour income shocks on unexpected stock returns.
As shocks to income and returns become more positively correlated, $\alpha^e$ falls – you hold less risky assets because when income is low, asset returns are also low, so the latter are less useful in smoothing consumption.

Conversely, if $\sigma_{eu} < 0$, then risky assets are a good hedge against unfavourable income shocks, so the hedging demand is positive (and then $\alpha^e > \alpha^r$).